Second stage of Israeli students competition, 2020.

Please try to write your solutions in English.

Duration: 4 hours

את השאלון ניתן לקחת איתך בסיום התחרות.

1. Let A be the number of ways to present 5780 as a sum of a finite sequence of non-decreasing positive integers, such that the 5th integer is 5. Let B be the number of ways to present 5755 as a sum of a sequence of length at most 9 of non-decreasing positive integers. Which of the two numbers is greater: A or $126 \cdot B$?

1'. Let A be the number of ways to present 5780 as a sum of a finite sequence of non-increasing positive integers, such that the 5th integer is 5. Let B be the number of ways to present 5755 as a sum of a sequence of length at most 9 of non-increasing positive integers. Which of the two numbers is greater: A or $126 \cdot B$?

2. Find the greatest real α for which the sequence $a_n = n^\alpha \int_0^\infty \left(\frac{\sin x}{x} \right)^n dx$ will converge, and compute $\lim a_n$ for that value of α.

3. A equilateral triangle with sides of length 100 is contained in the N-dimensional closed unit cube $[0,1]^N$. What is the minimal possible N?

4. Is it possible to find a bounded family of real numbers $a_{m,n}$ for $m,n \in \mathbb{N}$ such that $\lim_{n\to\infty} a_{m,n}$ is defined for any m, $\lim_{m\to\infty} a_{m,n}$ is defined for any n, but for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the sequence $a_{k,\sigma(k)}$ diverges?

5. Consider $n \times n$ matrices with real entries, given $n > 1$. A matrix N will be called **nilpotent** if N^n is a zero matrix. Let $\mathbf{S}$ the linear space of all $n \times n$ matrices which can be presented as a sum of nilpotent matrices, i. e. as $N_1 + N_2 + ... + N_k$ where all N_i are nilpotent.

(a) Find $\dim \mathbf{S}$.

(b) Find the minimal k such that any $M \in \mathbf{S}$ can be presented as $N_1 + N_2 + ... + N_k$ where all N_i are nilpotent.

6. Is there a polynomial of two variables $p(x,y)$ with real coefficients of degree 2, such that for any integer n there exists exactly one pair of integers x, y such that $p(x,y) = n$, **and** $p(x,y) \in \mathbb{Z}$ **for each** $x, y \in \mathbb{Z}$?

Good luck!