

**First stage of Israeli students competition, 2020.**

*Please try to write your solutions in English.*

*Duration: 4 hours*

*את השאלון ניתן לקחת איתך בסיום התחרות.*

1. Compute  $\lim_{N \rightarrow \infty} \frac{\sqrt[N]{\prod_{k=1}^N (k^k \cdot k!)}}{(N+1)!}$ .

2. What is the maximal possible value of  $\det \begin{pmatrix} * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \end{pmatrix}$ , if each \* is replaced

by 0 or 1? (Different stars might be replaced by different numbers.)

3. On an island, any two people who are not friends have precisely two common friends, and any two friends don't have common friends. Prove that each person has the same number of friends.

4. Find the greatest  $m \in \mathbb{R}$ , such that  $\int_0^5 f(x) dx \geq m$  for any convex function

$f: \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$f(0)=0, \quad f(1)=1^2, \quad f(2)=2^2, \quad f(3)=3^2, \quad f(4)=4^2, \quad f(5)=5^2.$$

Clarification: Function is called convex if the line segment between any two points on the graph of the function lies above or on the graph.

5. A cubic curve passes through 9 different points:

$$(0,1), (0,2), (0,5), (1,1), (1,3), (1,5), (3,7), (3,8), (3,y).$$

Find the value of  $y$ .

Reminder: a cubic curve is defined by an equation  $p(x,y)=0$  where  $p(x,y)$  is a polynomial of degree 3, i.e.  $p(x,y) = \sum_{i+j \leq 3} a_{i,j} x^i y^j$ , such that  $a_{0,3}, a_{1,2}, a_{2,1}, a_{3,0}$  are not all zeroes.

6. Is it possible to mark 16 distinct points, so that at distance precisely 1 from each marked point there will be precisely 10 marked points

(a) in  $\mathbb{R}^3$ ?

(b) in  $\mathbb{R}^4$ ?

**Good luck!**