Solutions: first stage of Israeli students competition, 2015.

1. Compute $\lim_{n\to\infty}\dfrac{\sqrt{n}}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}}$.

Answer. $\frac{1}{2}$.

First solution. By Stolz–Cesàro criterion criterion (similar to the more familiar l'Hôpital's rule but for sequences), if $\lim_{n\to\infty}\dfrac{\sqrt{n}-\sqrt{n-1}}{\frac{1}{\sqrt{n}}}$ exists, then it also gives an answer to the original question. But

$$\lim_{n\to\infty}\frac{\sqrt{n}-\sqrt{n-1}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\left(\sqrt{n}+\sqrt{n-1}\right)\left(\sqrt{n}-\sqrt{n-1}\right)}{\left(\sqrt{n}+\sqrt{n-1}\right)\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{1}{1+\sqrt{\frac{n-1}{n}}}=\frac{1}{1+1}=\frac{1}{2}$$

Second solution. Denote $L=\lim_{n\to\infty}\dfrac{\sqrt{n}}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}}$. Then

$$\frac{1}{L}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{\frac{1}{n}}}+\frac{1}{\sqrt{\frac{2}{n}}}+\frac{1}{\sqrt{\frac{3}{n}}}+\ldots+\frac{1}{\sqrt{1}}\right)=$$

$$=\int_0^1\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^1=2$$

Hence $L=\frac{1}{2}$.

2. Compute $\det\begin{pmatrix}1&2&2&3\\2&2&3&3\\2&3&3&4\\3&3&4&4\end{pmatrix}$.

First solution. We may subtract row 3 from row 4, and row 1 from row 2 without changing the determinant. We get

$$\det\begin{pmatrix}1&2&2&3\\2&2&3&3\\2&3&3&4\\3&3&4&4\end{pmatrix}=\det\begin{pmatrix}1&2&2&3\\1&0&1&0\\2&3&3&4\\1&0&1&0\end{pmatrix}$$

Now two of the rows (the second and the last) are the same, so det = 0.

Second solution. Notice that sum of two numbers in the middle is the same as the sum of two numbers at the ends of each row, so $\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$ is in the kernel, so determinant is zero.

3. Let *A*, *B*, *C*, *D* be points in 3-dimensional Euclidean space not in the same plane, such that the plane *ACB* is orthogonal to the plane *ACD*, and the plane *ABD* is orthogonal to the plane *CBD*. Prove that $\dfrac{\cos(\sphericalangle ACB)}{\cos(\sphericalangle ADB)}=\dfrac{\cos(\sphericalangle CBD)}{\cos(\sphericalangle CAD)}$.

Solution. Consider projection from *AD* to *BC* (each point *X* on *AD* is sent to the foot of perpendicular from *X* to *BC*). Length of each interval on *AD* is reduced by this projection by the same coefficient, which is $\cos\sphericalangle\left(\overrightarrow{AD},\overrightarrow{BC}\right)$.

This projection can be performed in two steps: first project point *X* on *AD* to point *Y* on *AC* (so that $XY\perp AC$, and then project point *Y* on *AC* to point *Z* on *BC*, so that $YZ\perp CD$. Indeed, planes *ACB*, *ACD* are given to be orthogonal, so *Y* is the projection of *X* to plane *ACB*, but *XY* is orthogonal to plane *ACB*, and to the line *BC*, therefore *X* and *Y* lie in the same plane, orthogonal to *BC*, hence projections of *X* and of *Y* to the line *BC* are in the same place. But now each projection is performed from a line to a line within the same plane, and each reduces distances by a coefficient which is cosine of an angle between two intersecting lines, so

$$\cos\sphericalangle\left(\overrightarrow{AD},\overrightarrow{BC}\right)=\cos\sphericalangle CAD\cdot\cos\sphericalangle ACB.$$

Now doing the same for the projection from *CD* to *AB*, using *BD* as intermediate line and the fact the plane *ABD* is orthogonal to the plane *CBD*, we conclude:

$$\cos\sphericalangle\left(\overrightarrow{CD},\overrightarrow{AB}\right)=\cos\sphericalangle ADB\cdot\cos\sphericalangle CBD.$$

Therefore $\cos\sphericalangle CAD\cdot\cos\sphericalangle ACB=\cos\sphericalangle ADB\cdot\cos\sphericalangle CBD$.
Q.E.D.

4. A finite number of polyhedrons of positive volume in 3-dimensional Euclidean space is given. Prove that one can mark a finite number of points in the same Euclidean space, so that strictly inside any two of given polyhedrons of equal volume, there will be the same number of marked points, and every given polyhedron will contain at least one point.

Solution. The number of marked points in each region (intersection of some polyhedrons and of complements to the other polyhedrons) can be denoted by a letter. Totally, there will be at most 2^N letters (some regions might be empty), where N is the number of polyhedrons, and each pair of polyhedrons of equal volume gives a linear equation. The equations are linear, and have integer coefficients, hence the solutions of the equations is linear subspace, spanned by rational vectors (which can be found by Gauss procedure). The equations have a positive solution: when to each region corresponds the volume of the region. It can be expressed as a linear combination of the rational basis vectors of the subspace, with possibly real coefficients. If the coefficients can be replaced by sufficiently close rational coefficients, we shall get a rational vector, which also has positive coordinates. Multiplying the vector by common denominator of the coordinates we get a vector with positive integer coordinates, which satisfies all the equations.
We can mark number of points in each region, according to the respective coordinates of the vector, and all conditions will be satisfied.

5. Prove that sum of digits of $2^{4^{1000001}}$ is greater than 1000000.

Solution. The last (units) digit is nonzero. It is not possible to have 3 consequent zeroes before the last digit, because the number consisting of 4 last digits has to be divisible by 16, but the last digit isn't divisible by 16. Let **u** be the number consisting of the last k digits. It is not possible to have $3k$ consequent zeroes before **u**. Otherwise **u** is divisible by $2^{4k} = 16^k >$ **u** $\neq 0$. So the number has at least $\log_4 d$ nonzero digits, where d is the total number of digits. The number of digits is

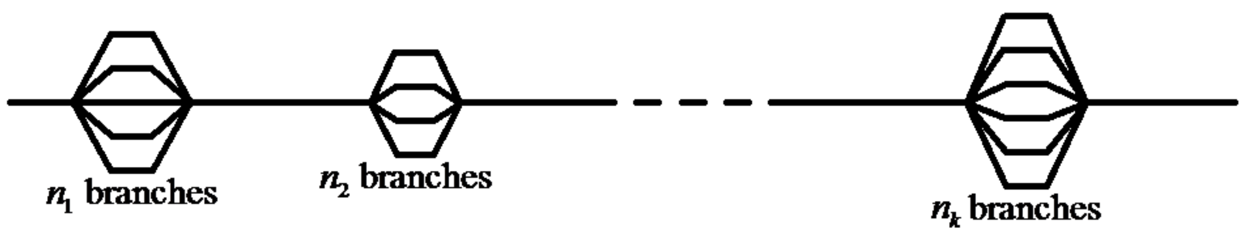
$$\log_{10}\left(2^{4^{1000001}}\right) = 4^{1000001} \cdot \log_{10} 2 \geq 4^{1000001} \cdot 0.3 > 4^{1000000}.$$

Remark. Then number 2 in this problem might be replaced by any even number which is not divisible by 5, or any odd number which is divisible by 5.

6. M cars move from left to right on a narrow road (they can't overtake each other, and cannot go backwards, all cars start at the left end and arrive to the right end). In k places, the road is split in parallel routes: first in n_1 parallel branches which are merged again, then in n_2 parallel branches, etc.

Each branch is long enough to contain any amount of cars.

For which M is it possible to reorder the cars in any possible way by the time they arrive to the right end of the road?



Answer. When $M \leq n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

Solution. The trajectory of each car is completely determined by the choice of branches, which has $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$ possibilities. If $M > N$, then by pigeonhole principle two of the cars have the same trajectory, hence they will be in the same order in the end as in the beginning.

Now we shall show that for $M \leq N$, any rearrangement is possible. We shall give each car a k-digit number. The last (least signifcant) digit of a number may be anything between 1 and n_1, the second least significant digit may be anything between 1 and n_2, and so on, the leading digit may be anything between 1 and n_k. If $n_i \geq 10$ we shall invent new digits.

The total number of possible numbers is N, so if $M \leq N$ we can assign each car a different number; we shall do it in such a way, that a car with a smaller number will be a car that should arrive earlier.
For each possible order of arrivals, such numbering is possible.

The driving will be according to the following three rules:

(a) All cars should arrive to the i'th split road and take its place on one of n_i parallel branches, before any car is allowed to continue to the next split road.

(b) The first to live i'th split road are the cars on the first parallel branch, the second are the cars of second parallel branch, the third are the cars of third parallel branch and so on.

(c) The decision for each car of which branch to take on the i'th split road is based on i'th least significant digit of its number (1 means first branch, 2 means second branch, and so on).

It is easy to see, that the less is the most significant digit of a number, the sooner the car will arrive, and given that first several digits of numbers of some two cars are the same, the car will arrive sooner if its next digit is less. Which means that the cars will arrive in order according to their numbers. Since numbers can be given in any possible order, the cars can arrive in any possible order.