**First stage of Israeli students competition, 2019.**
***Solutions***

**1.** Real invertible $n\times n$ matrices $A$ and $B$ are called **really different**, if for any nonzero vector $v\in\mathbb{R}^n$ the angle between $Av$ and $Bv$ is obtuse (זווית כהה). Find the greatest number $k$ such that there exist matrices $A_1, A_2, ..., A_k$ so that each 2 of them are really different

(a) for $n=3$, (b) for $n=4$.

Answers. (a) 2. (b) 5.

**Solution.** (a) It is easy to find two such matrices: if $A_2=-A_1$, then the angle between $A_1v$ and $A_2v$ is always $180^\circ$. It remains to prove that there can be no three really different matrices.

If there would be 3 different matrices, for two of them $A$ and $B$, we would have $\det A=\det B$. We claim that in this case there is a vector $v$ such that $Av$ and $Bv$ have precisely the same direction, i. e. the angle is zero. Consider the equation $Av=\lambda Bv$, where $\lambda$ is a number and $v$ is a nonzero vector. It is equivalent to the equation $B^{-1}Av=\lambda v$, which is the same as finding the eigenvalues and eigenvectors of $B^{-1}A$. The eigenvalues are the roots of the characteristic polynomial, which in our case is a real polynomial of degree 3. It has at least one real root. If this root is positive, we have found $v$ such that $Av$ and $Bv$ form a zero angle. If the first root $\lambda_1$ is negative, than the product of two other roots is also negative (because the product of all eigenvalues is the determinant, and $\det\left(B^{-1}A\right)>0$). Therefore $\lambda_2$ and $\lambda_3$ may not be complex conjugate, so they are real; precisely one of them is positive. So anyway, we have a positive eigenvalue.

**Second proof (topological).** We might replace each matrix $A$ by a continuous mappings from a unit sphere to itself: $v\mapsto\dfrac{Av}{|Av|}$. If $A$ and $B$ are really different, then the mappings of $A$ and $-B$ are homotopic to each other. For matrix of $\det>0$ the degree of the mapping is 1; for a matrix of $\det<0$ the degree of the mapping is $-1$. So, $A$ and $-B$ give the same degree of mapping in spheres, than they have the same sign of determinant; so for odd dimension such as 3, $\det A$ and $\det B$ should be of different signs, so there may be no more than 2 matrices.

(b) We cannot have more than 5 really different matrices. The left columns of those matrices should form obtuse angles with each other. We shall show that we cannot even have more than 5 nonzero vectors, such that angles in all pairs are obtuse.

Assume we have vectors $v_1,...,v_k \in \mathbb{R}^n$ such that an angle in each pair is obtuse. Assume that $v_k = (0,0,...,0,1)$. Then the last coordinate in all other vectors is strictly negative. Let $u_i$ be a vector obtained from $v_i$ by erasing the last coordinate, for $i = 1,...,k-1$. Then $\langle u_i, u_j \rangle < \langle v_i, v_j \rangle < 0$. So, if there are $k$ vectors with obtuse angles in $\mathbb{R}^n$, there are $k-1$ such vectors in $\mathbb{R}^{n-1}$ as well. In $\mathbb{R}^1$ there are no more than two such vectors, so in $\mathbb{R}^4$ no more than 5 vectors, so we cannot have more than 5 really different matrices.

Now we shall explain how to construct 5 such matrices. First we construct 5 vectors $v_1,...,v_5$, which form obtuse angles; vertices of a regular simplex centered at the origin will do. Than we stretch them, until all the vectors are unit vectors. Now, we may say that $\mathbb{R}^4 = \mathbb{H}$, the quaternions. On the quaternions a product is defined; each unit vector gives a unit quaternion, and product by this quaternion (from the left) induces a geometric rotation of $\mathbb{R}^4$, so we get 5 orthogonal $4\times 4$ matrices $A_1,...,A_5$. For each vector $w$, the vectors $A_1w, A_2w,...,A_5w$, are the same as quaternions $v_1w,...,v_5w$, which have the same angles between each other as $v_1,...,v_5$ which are all obtuse.

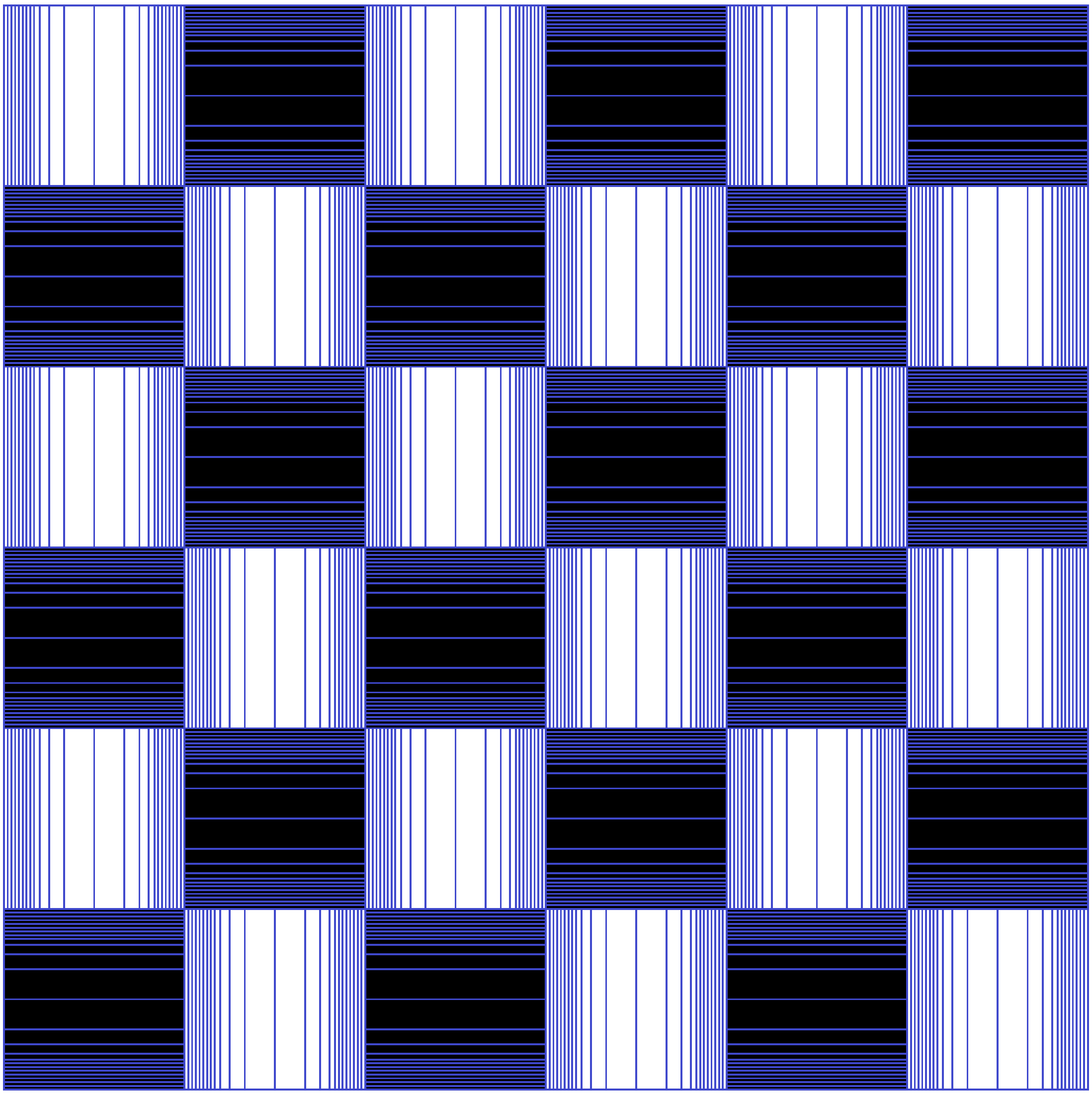
**2.** Let $F_1, F_2,...,$ be an infinite sequence of closed subsets of $\mathbb{R}^2$, such that the intersection of any three different sets in this sequence is empty, and for any $i$ for each two points $p,q \in F_i$, the distance between $p$ and $q$ is less than 1. Might it be that $\bigcup_{n=1}^{\infty} F_n = \mathbb{R}^2$ ?

Answer. Yes, it might.

**Solution.** First, we shall show how to cover the rectangle $(0,1)\times[0,1]$ (open on one coordinate, closed on another) by rectangles with no triple intersection. Consider rectangles $(0,1)\times\left[\frac{1}{n+1},\frac{1}{n}\right]$ for $n=2,3,\ldots$ and then also rectangles $(0,1)\times\left[1-\frac{1}{n},1-\frac{1}{n+1}\right]$ $n=2,3,\ldots$

Divide the plane into unit squares, with checker-board black and white coloring. The black cells, including their vertical borders (but not vertices) will be covered by rectangles according to the translations of the above construction. The white cells including their horizontal borders will be covered by translation of a similar construction, rotated by $90^\circ$. The Only uncovered parts are the integer vertices; these are $\aleph_0$ separate closed sets. We have $\aleph_0$ cells, each covered by $\aleph_0$ rectangles, so together with the points we get $\aleph_0$ closed sets, and there are no triple intersections.

A diagonal of each rectangle is less than 1.5, so if we reduce the picture 1.5 times, all the conditions will be satisfied.



**3.** Two regular pentagons, $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$, have the same circumcircle. For which $n$, we can claim that any polynomial $p$ of degree $n$ on $\mathbb{R}^2$ satisfies $\sum_{i=1}^{5} p(A_i) = \sum_{i=1}^{5} p(B_i)$.

Answer. For $n < 5$.

**Solution.** Let us shift the origin to the center of the circumcircle (translation does not alter the degree). We may dilate the plane so that the radius of the circle would be one. The polynomial of degree less than 4 in $x$ and y might be expressed as a polynomial of degree less than 4 in $z = x + iy$ and $\overline{z} = x - iy$. On the unit circle, we may simplify using the relation $z \cdot \overline{z} = 1$, so we will get a polynomial $p(z) = \sum_{k=-4}^{4} a_k z^k$ .

Denote $\xi = e^{2\pi i/5}$. We may construct a new polynomial:

$$q(z) = \sum_{j=0}^{4} p\left(\xi^j z\right)$$

It is easy to see that things cancel out, and we get:

$$q(z) = \sum_{j=0}^{4} \sum_{k=-4}^{4} a_k \xi^{jk} z^k = 5a_0$$

So, $q$ is constant on the circle, which means precisely that the sum of $p$ over the vertices of pentagon is the same for both pentagons.

For degree 5, it is easy to construct a counterexample. Take tangent lines to the circle at vertices $A_1, A_2, A_3, A_4, A_5$. Those lines may be defined by linear equations $\ell_1, ..., \ell_5$; the signs might be chosen in such a way that $\ell_i$ is strictly positive on the circle (except $A_i$ on which it is 0). The product $\ell_1 \cdot \ell_2 \cdot \ell_3 \cdot \ell_4 \cdot \ell_5$ gives $\sum_{i=1}^{5} p(A_i) = 0 < \sum_{i=1}^{5} p(B_i)$.

For higher degrees there are much more counterexamples, we can always add the powers of $x^2 + y^2$, or multiply counterexample by a power of $x^2 + y^2$.

**Second solution.** We can rotate the plane around the center of the circumcircle by $\frac{2\pi}{5}$ several; we get 5 different polynomials so we can sum them up to get a new polynomial $q$ of the same degree. If degree of $p$ was at most 4, then $q$ is of degree at most 4.

The question is, whether $q$ a function of radius. Assume that $q$ is not constant on the circle. On a circle it achieves its maximal value $m$ at least in 5 points, and it has zero tangent derivative at those points. The level lines $q = m$ are tangent to the circle in at least 5 points, so the number of intersection points, computed with multiplicity is 10 at least. The level set $q = m$ is an algebraic curve of degree 4 at most; the circle is an algebraic curve of degree 2 precisely, so by Bezout theorem we are allowed no more than 8 intersection points with multiplicity. The only possible explanation is that the two curves have an infinite number of intersection points, but the circle is irreducible, so it is contained in the level set, so $q$ is in the level set.

The counterexample is the same as in the first solution.

**4.** In a torsion free group $G$, for any $a, b \in G$ the identity $a^n b^n = (ab)^n$ is satisfied. Prove that $G$ is abelian.

**Solution.** We may cancel and $a$ on the left and a $b$ on the right:

$$a^{n-1} b^{n-1} = (ba)^{n-1}$$

Also,

$$x^{n^2} y^{n^2} = (x^n)^n (y^n)^n = (x^n y^n)^n = (xy)^{n^2}$$

By cancelation $x^{n^2-1} y^{n^2-1} = (yx)^{n^2-1}$.

On the other hand

$$(x^{n+1} y^{n+1})^{n-1} = (y^{n+1})^{n-1} (x^{n+1})^{n-1} = y^{n^2-1} x^{n^2-1} = (xy)^{n^2-1}$$

$$1 = (x^{n+1} y^{n+1})^{n-1} \left((y^{-1} x^{-1})^{n+1}\right)^{n-1} = \left((y^{-1} x^{-1})^{n+1} x^{n+1} y^{n+1}\right)^{n-1}$$

So the expression in the brackets must be 1, as the group has no torsion.

$$1=\left(\left(y^{-1}x^{-1}\right)^{n+1}x^{n+1}y^{n+1}\right)^{n-1}$$

$$1=\left(y^{-1}x^{-1}\right)^{n+1}x^{n+1}y^{n+1}$$

$$(xy)^{n+1}=x^{n+1}y^{n+1}$$

By cancelation $$(yx)^{n}=x^{n}y^{n}=(xy)^{n}$$

$$1=(xy)^{n}\left(x^{-1}y^{-1}\right)^{n}=\left(xyx^{-1}y^{-1}\right)^{n}$$

Since there is no torsion

$$1=xyx^{-1}y^{-1}$$

Which is equivalent to the statement.

**5.** Show that there is no more than $\aleph_0$ positive numbers $\alpha$ which can be expressed as a limit of $\{a^n\}$ - a fractional part of a geometric progression.

**Solution.** It is enough to show that there are no more than $\aleph_0$ positive values of $a$ for $1<a<M$ (as for $a\le 1$ it is clear the limit is zero). Then we might take a union for all natural $M$, and a union of $\aleph_0$ sets of at most $\aleph_0$ elements is at most $\aleph_0$. We may safely assume that $M>100$.

Take $\varepsilon=\frac{1}{4M}$. For any integer $k\in[0,4M-2]$, construct an interval $I_k=[k\varepsilon,(k+2)\varepsilon)$. The intervals cover with overlaps the interval $[0,1)$, which is the space of conceivable fractional parts. So if $a^n\to\alpha$ then there is a $k$ (either such that $\alpha$ is in the interior of $I_k$, or one of the two extreme intervals if $\alpha$ is 0 or 1), such that for each $n$ large enough $\{a^n\}\in I_k$.

After $k$ has been chosen, consider the family of intervals $J_m=m+I_k$, for all $m>M$. So, we may say that for $n$ large enough, the sequence $\{a^n\}$

belongs to $\bigcup_{m=M+1}^{\infty} J_m$. We shall consider only $n$ large enough and disregard the small elements of the geometric sequence, so $\{a^n \mid n > s\} \subset \bigcup_{m=M+1}^{\infty} J_m$.

Let us fix $m_0$ and $n_0$ such that $a^{n_0} \in J_{m_0}$, and for each $n \geq n_0$ we also have $\{a^n\} \in I_k$. There are $\aleph_0$ different options for the possible values of $m_0$ and $n_0$, so it suffices to show that for each given $m_0$ and $n_0$ there are at most $\aleph_0$ different values of $a$.

We do not have to choose the smallest possible $n_0$, so we may safely assume that $n_0 > 10$.

If $a^n \in J_m$, we may ask ourselves what are the possible values of $a^{n+1}$. Let us denote $x = a^n$, then $a^{n+1} = x^{\frac{n+1}{n}}$.

But $\left(x^{\frac{n+1}{n}}\right)' = \frac{n+1}{n} \cdot x^{\frac{1}{n}} = \frac{n+1}{n} \cdot a < \frac{11}{10} M$, assuming $a < M$. So the interval $J_m$ of length $2\varepsilon$ is stretched (by the transformation $x \mapsto x^{\frac{n+1}{n}}$) to an interval of length at most $\frac{11}{10} M \cdot 2\varepsilon = \frac{11}{20}$. So if $x \in J_m$ only for a unique natural $t$ the interval of all possible $x^{\frac{n+1}{n}}$ may intersect $J_t$, as distance between the two closest points from $J_t$ and $J_{t+1}$ is $1 - 2\varepsilon \geq 0.8 > \frac{11}{20}$.

So, if $a$ satisfies the conditions of the problem, and $a^{n_0} \in J_{m_0}$ where $n_0, m_0$ are chosen as we assumed, then there should be a unique $m_1$ such that $a^{n_0+1} \in J_{m_1}$, and in the same way there is a unique $m_2$ such that $a^{n_0+2} \in J_{m_2}$ and so on, by induction repeating the same argument for $n = n_0 + \nu$ and $m = m_\nu$ we get by induction that there is a unique $m_{\nu+1}$ such that $a^{n_0+\nu+1} \in J_{m_{\nu+1}}$.

If $a^{n_0+\nu+1} \in J_{m_{\nu+1}}$, we can conclude that $a$ belongs to a very small interval, since $f(a) = a^n$, then $f'(a) = na^{n-1} > n$ for $a > 1$, so the length of each interval is stretched at least $n$ times. But the length of $J_{m_{\nu+1}}$ is $2\varepsilon$, so the length of interval it allows for $a$ is at most $\frac{1}{n_0+\nu+1}$. This tends to 0, so $a$ is defined uniquely.

So after choice among at most $\aleph_0$ options of values for natural numbers $M$, $n_0$ and $m_0$, we get no more than one possible value of $\alpha$. So we get only at most $\aleph_0$ values of $\alpha$.

**Remark.** It is interesting whether there exists any such $\alpha$ which is not 0 or 1.

**6.** Assume $p$ is a prime, $k_0, k_1, ..., k_{p-1}$ are nonnegative integers such that $k_0 + k_1 + ... + k_{p-1} < p - 1$. Prove that $\sum_{\sigma \in S_p} \prod_{i=0}^{p-1} \binom{i - \sum_{j=0}^{i-1} k_{\sigma(j)}}{k_{\sigma(i)}}$.

**Solution.** Let us give a combinatorial interpretation of the expression

$$f(k_0, k_1, ..., k_{p-1}) = \prod_{i=0}^{p-1} \binom{i - \sum_{j=0}^{i-1} k_j}{k_i}.$$

Ayala and Barvaz are playing a game, **Fish Sorting**:

Ayala has $p$ different (numbered) fishes, and Barvaz has $p+1$ buckets. The bucket number zero is special (made of gold), all other buckets are also numbered.

The game consists of $p+1$ turns, numbered $0, 1, 2, ..., p$.

On turn $i$, Barvaz puts bucket $i$ on the table (starting with the golden bucket); then Ayala puts fish number $i$ in one of the buckets which are on the table.

At the last turn, Barvaz puts bucket $p$ on the table, and then the game is over since Ayala has no more fish.

**Lemma.** $f\left(k_0,...,k_{p-1}\right)$ is the number of ways the game could go so that in the end there are $k_i$ fishes in the bucket $p-i$.

**Proof.** Go over the buckets in reverse order and choose the turns at which Ayala put a fish in it. In the bucket $p$ there should be $k_0$ fishes, that is $\binom{0}{k_0}$ options (Ayala didn't have a chance to put a fish in the last bucket).

In the bucket $p-1$ Ayala could put just the last fish, so we have $\binom{1-k_0}{k_1}$ options. In the bucket $p-m$ we have $\binom{m-k_0-...-k_{m-1}}{k_m}$ options, since we have to choose among the last $m$ fishes, setting aside fishes used in the other buckets already

The product of the numbers is precisely the expression.

***Note that the operator of choosing a bucket and putting a fish in it, reminds the Leibniz rule for derivative of a long product, we choose each time which term to derive. We will attempt to interpret the game as an algebraic operator. Ayala's turns would be derivations.***

Let $\mathbb{F}_p$ be the field of $p$ elements. Let $R$ be the **non-commutative** ring over $\mathbb{F}_p$ generated by the infinite set of $d^i f, d^i g$ for $i \geq 0$ (here $d^i$ means derivative of order $i$). Define $d: R \to R$ by $d\left(d^i f\right) = d^{i+1} f$ and $d\left(d^i g\right) = d^{i+1} g$, and the Leibniz rule: $d\left(AB\right) = \left(dA\right) \cdot B + A \cdot dB$.

On monomials $d$ is well defined: $d\left(a_1 a_2 ... a_k\right) = \sum_{i=1}^{k} a_1 a_2 ... a_{i-1}\left(da_i\right) a_{i+1} ... a_k$,

and thus $d$ is defined on polynomials; $d$ satisfies Leibniz rule for products of elements in $R$.

A monomial $\left(d^{k_0} f\right)\cdot\left(d^{k_1} f\right)\cdot\ldots\cdot\left(d^{k_\ell} f\right)\cdot\left(d^{k} g\right)$ represents a state in which each term is a bucket, the number of derivatives in each term is the number of fishes in that bucket.

After each turn we will have the sum over monomials representing the states we can get.

So we get a new meaning for the game: Ayala and Barvaz perform a computation together.

At first Barvaz writes the expression $E_0 = g$. Each time, Ayala computes $E_i$ to get $dE_i$ and then Barvaz multiplies $dE_i$ by $f$ on the left to get $E_{i+1} = f\cdot dE_i$. In the end, they compute $E_p = \left(f\cdot d\right)^p g$.

Then , $f\left(k_0, k_1, \ldots, k_{p-1}\right)$ is the coefficient in $E_p$ of the monomial

$$\left(d^{k_0} f\right)\cdot\left(d^{k_1} f\right)\cdot\ldots\cdot\left(d^{k_{p-1}} f\right)\cdot\left(d^{\,p-k_0-k_1-\ldots-k_{p-1}} g\right).$$

The element which hopefully vanishes is the sum of the coefficients $\left(d^{k_0} f\right)\cdot\left(d^{k_1} f\right)\cdot\ldots\cdot\left(d^{k_{p-1}} f\right)\cdot\left(d^{\,p-k_0-k_1-\ldots-k_{p-1}} g\right)$ with all its permutations in $\left(f\cdot d\right)^p g$. Note that we count each coefficient many times – we allow permutations that does not change the sequence $k_i$.

Thus what we actually want to calculate is some coefficient (which is product of several factorials) times the coefficient of $\left(d^{k_0} f\right)\cdot\left(d^{k_1} f\right)\cdot\ldots\cdot\left(d^{k_{p-1}} f\right)\cdot\left(d^{\,p-k_0-k_1-\ldots-k_{p-1}} g\right)$ in $\left(fd\right)^p g$, but when working in the commutative ring $\tilde{R}$, which is defined just like $R$ - but commutative.

Let $H$ be a commutative ring constructed by adding the formal variables $d^i h$ to $\tilde{R}$.

Consider the differential operator $D = fd$. Note that $D$ satisfies the Leibniz rule – because $d$ satisfies it.

Let $\delta$ be an operator satisfying the Leibniz rule. We claim that $\delta^p$ also satisfies the Leibniz rule.

Indeed one can see by induction that $\delta^k(AB)=\sum_{i=0}^{k}\binom{k}{i}\delta^i A\cdot\delta^{k-i}B$.

We can expand $D^p=\sum_{i=0}^{p}F_i\cdot d^i$, where $F_i$ is a polynomial in different $d^k f$. We want to compute $D^p g=\sum_{i=0}^{p}F_i\cdot d^i g$.

**Lemma.** $F_i=0$ for $i\neq 1,p$.

**Proof.** Consider the equality $D^p(gh)=\left(D^p g\right)\cdot h+g\cdot D^p h$. It follows that $\sum_{i=0}^{p}F_i\cdot\left(d^i(gh)-\left(d^i g\right)h-\left(d^i h\right)g\right)=0$.

Note that each summand uses its own unique set of monomials; the $i$-th summand uses monomials with total $i$ number of $d$ to $g,h$. It follows that for every $i\neq 1,p$ we have $F_i\cdot\left(d^i(gh)-\left(d^i g\right)h-\left(d^i h\right)g\right)=0$.

Since $d^i(gh)-\left(d^i g\right)h-\left(d^i h\right)g\neq 0$ for $i\neq 1,p$ it follows that $F_i=0$ for $i\neq 1,p$ as desired.

Since the coefficient of $d^{k_0}f\cdot d^{k_1}f\cdot\ldots\cdot d^{k_\ell}f$ in $D^p g$ is the coefficient of $d^{k_0}f\cdot d^{k_1}f\cdot\ldots\cdot d^{k_\ell}f$ in $F_{p-k_0-k_1-\ldots-k_{p-1}}$ it follows that it vanishes for $p-k_0-k_1-\ldots-k_{p-1}\neq p,1,$ that is, for $0<k_0+k_1+\ldots+k_{p-1}\leq p-2$.
The case that remains is $k_0+k_1+\ldots+k_{p-1}=0$. Then all summands in the sum are equal, and their number is $p$!