First stage of Israeli students competition, 2019.

Please try to write your solutions in English.

Duration: 4 hours

את השאלון ניתן לקחת איתך בסיום התחרות.

1. Real invertible $n \times n$ matrices A and B are called **really different**, if for any nonzero vector $v \in \mathbb{R}^n$ the angle between Av and Bv is obtuse (זווית קהה). Find the greatest number k such that there exist matrices $A_1, A_2, ..., A_k$ so that each two of them are really different

(a) for $n = 3$, (b) for $n = 4$.

2. Let $F_1, F_2, ...,$ be an infinite sequence of closed subsets of $\mathbb{R}^2$, such that the intersection of any three different sets in this sequence is empty, and for any i for each two points $p, q \in F_i$, the distance between p and q is less than 1. Might it be that $\bigcup_{n=1}^{\infty} F_n = \mathbb{R}^2$?

3. For which n, it is true that for any polynomial p of degree n on $\mathbb{R}^2$ and for any two regular pentagons, $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ with the same circumcircle the identity $\sum_{i=1}^{5} p(A_i) = \sum_{i=1}^{5} p(B_i)$ holds?

4. In a torsion free group G, for any $a, b \in G$ the identity $a^n b^n = (ab)^n$ is satisfied. Prove that G is abelian.

Reminders. A group is called ***torsion free***, if for any natural $n \geq 1$ and for any $g \in G$, if $g^n = 1$ then $g = 1$. A group is called ***abelian***, if $ab = ba$ for any $a, b \in G$.

5. Show that there is no more than $\aleph_0$ positive numbers α which can be expressed as a limit of $\{a^n\}$ for $a > 0$ (a fractional part of a geometric progression).

6. Assume p is a prime and $k_0, k_1, ..., k_{p-1}$ are nonnegative integers such that $k_0 + k_1 + ... + k_{p-1} < p - 1$. Prove that $\sum_{\sigma \in S_p} \prod_{i=0}^{p-1} \binom{i - \sum_{j=0}^{i-1} k_{\sigma(j)}}{k_{\sigma(i)}}$ is divisible by p.

Good luck!