**Solutions: first stage of Israeli students competition, 2019.**

**1.** Let $A,B$ be two orthogonal $n\times n$ matrices with real entries. What is the maximal possible value of $\det(A+B)$?

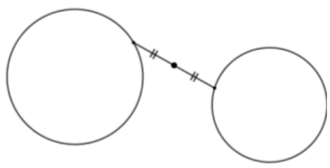
**First solution.** Let $c_{i,0}$ be the columns of $A$, and $c_{i,1}$ be the columns of $B$. Then since determinant is multi-linear in columns, then

$$\det(A+B)=\sum_{k_1,k_2,\dots,k_n\in\{0,1\}} Vol\left(v_{1,k_1},v_{2,k_2},\dots,v_{n,k_n}\right),$$

where $Vol(\dots)$ is the oriented volume of parallelepiped spanned by the given vectors. We have $2^n$ summands, each one is a volume of parallelepiped with unit side lengths, so one at most; hence the value is at most $2^n$. If $A=B=$the unit matrix, this value is achieved.

**Second solution.** Each column of $A+B$ is a column of $A$ plus a column of $B$. Column of $A$ and $B$ are unit vectors, so by triangle inequality a column $A+B$ is of length 2 at most, and $\det(A+B)$ is a volume of a parallelepiped with sides of length 2 at most, so the volume is at most $2^n$. The example is the same as in previous solution.

**2.** Two circles are given in plane: circle $\alpha$ of radius $a$, and circle $\beta$ of radius $b$. Consider all midpoints of intervals, connecting a point on $\alpha$ and a point on $\beta$. What is the area of the set of all such midpoints?



Answer. $\pi ab$.

**Solution.** If we write each point as a vector, we consider points of type $\frac{A+B}{2}$, where $A\in\alpha$ , $B\in\beta$. So, if we shift one of the circles be some vector $v$, we each average is shifted by $\frac{v}{2}$, but the area of the locus remains the same. So WLOG both circles are centered at the origin. In other words, we take points $\frac{A+B}{2}$ such that $|A|=a$ and $|B|=b$. Then by triangle inequality, $a-b=|A|-|-B|\le|A-(-B)|=|A+B|\le a+b$. Hence $\frac{|a-b|}{2}\le\left|\frac{A+B}{2}\right|\le\frac{a+b}{2}$. Both bounds can be reached (the low bound when the vector have opposite direction, the upper bound when they have the same direction, and all the values between are also obtained by continuity. Since the shape has rotational symmetry (vectors may be rotated around zero), the locus is the annulus (טבעת) between two circles of radii $\frac{a+b}{2}$ and $\frac{|a-b|}{2}$, and its area is the difference between two circles: $S=\pi\left(\frac{a+b}{2}\right)^2-\pi\left(\frac{a-b}{2}\right)^2=\frac{\pi}{4}\left((a+b)^2-(a-b)^2\right)=\pi ab$.

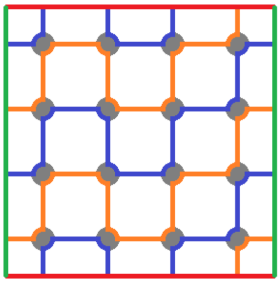
**3.** Compute $\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{2n}\left(n-|n-k|\right)\cdot\sqrt[n]{e^k}}{n^2}$.

Answer. $(e-1)^2$.

**Solution.** Consider a function of two variables $f(x,y)=e^{x+y}$. We want to integrate it on the square $[0,1]^2$. So, sample it at points $\left(\frac{\ell}{n},\frac{m}{n}\right)$ where $\ell\in\{1,2,3,\dots,n\}$ and $m=\{0,1,2,\dots,n-1\}$. This lattice becomes dense as $n\to\infty$ so the sum becomes an approximation of the integral. Notice that the value of the function depends only on $x+y$ so we need to count the number of points with given $k=\ell+m$ which is precisely $n-|n-k|$. The area of each cell is $\frac{1}{n^2}$. Therefore

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{2n}\left(n-|n-k|\right)\cdot\sqrt[n]{e^k}}{n^2}=\lim_{n\to\infty}\frac{\sum_{\ell=1}^{n}\sum_{m=0}^{n-1}e^{\frac{\ell+m}{n}}}{n^2}=\int_0^1\int_0^1 e^{x+y}dxdy=\left(\int_0^1 e^x dx\right)^2=(e-1)^2$$

**4.** Consider a four dimensional cube as a graph: vertices of the cube are vertices of the graph, and edges are edges of the graph. Is it possible to find two Hamilton cycles, which don't have any common edge?



**Solution.** Yes. To find the solution, it helps if you can visualize the 4-dimensional cube. Notice that two dimensional cube is a square, which is (topologically) a circle. So a 4-dimensional cube can be naturaly imbedded into a torus (which is a Cartesian squares of a circle). Torus might be visualized as a rectangle, in which the opposite sides are glued to each other by parallel shifting. After the 4-dimensional cube is visualized in that way, it is easy to find such Hamiltonian cycle (here another topological idea is useful: on torus there is no Jordan theorem, you can find a close loop which does not separate the torus into two parts).

**5.** Let $F_1, F_2, F_3, F_4, \ldots$ be an infinite sequence of pairwise disjoint non-empty closed sub-sets of $\mathbb{R}$. Might it be that $\bigcup_{n=1}^{\infty} F_n = \mathbb{R}$?

**Answer.** No.

**Solution.** Take $a_1 \notin F_1$ and $b_1 \in F_1$. Let $J_1$ be the closed interval with endpoints $a_1$ and $b_1$, and $G_1 = J_1 \cup F_1$.

$$c_1=\begin{cases}\sup(G_1) & b_1<a_1\\ \inf(G_1) & a_1<b_1\end{cases}$$

Then $c_1 \in F_1$ since $F_1$ is closed. Let $I_1$ be the closed interval with endpoints $a_1$ and $c_1$. Then, on $I_1$ there is just one representative, and that is one of the endpoints. We shall define a sequence of nested closed intervals $I_k$ inductively. If $F_k$ is disjoint from $I_{k-1}$, we define $I_k = I_{k-1}$.

Otherwise there is a point $b_k \in I_{k-1} \bigcap F_k$. By construction, one of the endpoints of $I_{k-1}$, we shall call it $a_k$, is a point of $F_j$ for some $j < k$ and it is the only point of $b_k \in I_{k-1} \bigcap F_j$, and $a_k$ is the only point of $I_{k-1} \bigcap F_j$. Obviously, $a_k \notin F_k$ since $F_k$ and $F_j$ are disjoint. A closed interval with the endpoints $a_k$ and $b_k$ will be denoted $J_k$. Denote also $G_k = J_k \bigcap F_k$. Take

$$c_k = \begin{cases} \sup\left(G_k\right) & b_k < a_k \\ \inf\left(G_k\right) & a_k < b_k \end{cases}$$

Notice that now only one point of $I_k$ is covered by $F_1 \bigcup F_2 \bigcup \ldots \bigcup F_k$, and it is one of its endpoints.

Then $c_k \notin F_k$ and all the points between $a_k$ and $c_k$ don't belong to $F_k$ or to any $F_i$ for $i < k$. Let $I_k$ be the close interval with endpoints $\frac{a_k + c_k}{2}, c_k$.

From this inductive construction, we get a sequence of nested closed intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$. By the properties of real numbers, there is a point which belongs to the intersection of all those interval, $p \in \bigcap_{k=1}^{\infty} I_k$.

If all real numbers are covered by $\bigcup_{i=1}^{\infty} F_i$, the intervals are reduced infinite number of time. So for each natural $m$ there is $k > m$, such that $I_k$ has just one point from $\bigcup_{i=1}^{k} F_i$ and that is $F_k$, so $F_m$ is disjoint from $I_k$, hence $p \notin F_m$ for all $m$. So $p \notin \bigcup_{i=1}^{\infty} F_i$.

**6.** Let $0 < a_0 < a_1 < ... < a_n$. Prove that

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + ... + a_n \cos(nx)$$

has precisely $n$ roots on the interval $[0, \pi]$.

**Solution.** We shall use the following fact:

**Lemma.** If $0 < a_0 < a_1 < ... < a_n$, the polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ has roots only in the open unit disk $\{|z| < 1\}$.

From the lemma follows, by the argument principle, when $z$ travels around a circle $|z| = 1$, the value of $p(z)$ goes around 0 precisely $n$ times, so it $p(z)$ meets the positive ray at least $n$ times, and meets the negative ray at least $n$ times.

So $\text{Re}(p(z))$ has at least $2n$ zeroes on the circle $|z| = 1$.

We shall show that it has at least $2n$ zeroes. Let $q(x, y) = \text{Re}(p(x + iy))$.

Multiplying $q(x, y) \cdot q(x, -y)$ we get a polynomial which is even in $y$, and of degree $2n$. We may substitute $y = \sqrt{1 - x^2}$. Then we get a polynomial $Q(x) = q\left(x, \sqrt{1 - x^2}\right) \cdot q\left(x, -\sqrt{1 - x^2}\right)$ in $x$ of degree $2n$, since the root appears always in even power. Each root appears twice in $Q$, since if $q(x, y) = 0$ then $q(x, -y) = 0$ as well, and $x = \pm 1$ are not roots of $Q$, since $p(\pm 1)$ has nonzero real part. So $Q$ has at most $n$ distinct roots, and $\text{Re}(p(z))$ has at most $2n$ zeroes on the circle $|z| = 1$.

So $\text{Re}(p(z))$ has precisely $2n$ zeroes on the circle $|z| = 1$.

If we take $z = e^{it}$, we could reformulate what we've proven by saying that $f(t)$, where $f$ is as in the problem statement, has precisely $n$ zeroes in $[-\pi, \pi]$, none of which are 0 or $\pm\pi$. Notice that $f$ is even; hence precisely half of those roots are in $(0, \pi)$.

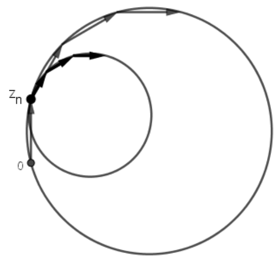
So, it remains to prove the lemma. We shall present two proofs:

**First proof of the lemma.** We should prove that

$$a_n z^n + \ldots + a_2 z^2 + a_1 z + a_0 \neq 0, \text{ if} |z| \geq 1.$$

Consider the complex numbers $a_n z^n, \ldots, a_2 z^2, a_1 z, a_0$: the argument is an arithmetic sequence, and the absolute value is decreased each time.

Notice that if the arguments in the sequence $z_n, z_{n-1}, \ldots, z_0$ would be an arithmetic sequence, and the absolute values would be the same, then the sequence of partial sums $0, z_n, z_n + z_{n-1}, z_n + z_{n-1} + z_{n-2}, \ldots$ in complex plane would be a on circle, since each four consequent elements would be an isosceles trapezoid (טרפשו"ש).




If we reduce simultaneously the absolute values of $z_{n-1}, z_{n-2}, \ldots, z_1, z_0$, in other words we take arguments and $|z_n|$ the same as before and $|z_n| > |z_{n-1}| = \ldots = |z_0|$, we get that all partial sums except $0$ and $z_n$, namely $z_n + z_{n-1}, z_n + z_{n-1} + z_{n-2}, \ldots$ were pulled homothetically towards $z_n$, and now they are on a smaller circle, tangent to the original circle at $z_n$ (where $\frac{|z_{n-1}|}{|z_n|}$ is the ratio of homothety).

Now we reduce $|z_{n-2}| = \ldots = |z_0|$ and notice that $z_n + z_{n-1} + z_{n-2}, \ldots$ all belong to an even smaller circle etc. So in the end the sum is inside the sequence of circles, each containing the next. The original circle is passing through zero, the second circle is tangent to it at $z_n \neq 0$, so the sum is definitely nonzero. This proves the lemma.

**Second proof of the lemma.** Multiply the polynomial by $z-1$.

$$(z-1)\left(a_n z^n + \ldots + a_2 z^2 + a_1 z + a_0\right) = a_n z^n - b_{n-1} z^{n-1} - \ldots - b_2 z^2 - b_1 z - b_0,$$

Notice that as $0 < a_0 < a_1 < \ldots < a_n$ we see that $b_k > 0$ for each $k$, and the leading coefficient $a_n = b_{n-1} + \ldots + b_2 + b_1 + b_0$ since 1 is a root of this new polynomial. If $|z| \geq 1$ then

$$|a_n z^n| \geq |b_{n-1} z^{n-1}| + \ldots + |b_2 z^2| + |b_1 z| + |b_0| \geq |b_{n-1} z^{n-1} + \ldots + b_0|$$

Equality in this inequality might happen only if $z = 1$, which is indeed a root of a new polynomial, but it wasn't a root of an original polynomial.