Monsky's theorem

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August 22, 2024

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- History.

The $p-$adic valuation.

Let p be a prime. The $p-$adic valuation is the function

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$$

satisfying:

- $v_p(p^n \frac{a}{b}) = n$ if $p \nmid a, b$.
- $v_p(0) = \infty$.

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Properties:

- $|0|_p = 0$ and for all other x, $|x|_p > 0$.
- $|xy|_p = |x|_p|y|_p$.
- $|x+y|_p \leqslant |x|_p + |y|_p$.

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Theorem (Chevalley)

$|\cdot|_p$ extends to a function $|\cdot|'_p : \mathbb{R} \to \mathbb{R}$ with the same properties.

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- $|1|_2' = ?$
- $|2|_2' = ?$
- $|\frac{1}{2}|_2' = ?$
  $|-1|_2' = ?$
- Important for later: For an integer n,

$$|\frac{1}{n}|_2' > 1$$

if and only if n is even.

Recall that:

- A *dissection* of a polygon R into triangles is a finite collection of triangles in R which cover R and their interiors are disjoint.
- A *triangulation* of R is a dissection into triangles such that now vertex in the interior of R lies on an edge of a triangle.

Consider a triangulation of a polygon R. Suppose we color each vertex of the triangulation (including those of R) by blue, red and yellow.
We say that an edge if *bicolored* if its two vertices have two different colors. We say that a triangle is tricolored if its vertices are of the three different colors.

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Theorem (Sperner's lemma)

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Proof.

An exercise! □

Step 1: An auxiliary coloring of the plane.
Define the following sets:

$$B = \{(x, y)|\ |x|_2' < 1, |y|_2' < 1\}$$

$$R = \{(x, y)|\ |x|_2' \geqslant 1, |x|_2' \geqslant |y|_2'\}$$

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Observation: B, R, Y partition $\mathbb{R}^2$.

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2. Every line in the plane is colored by at most 2 colors.
3. Let T be a tricolored triangle. Then $|\text{Area}(T)|_2' > 1$.

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- The edge from $(1,0)$ to $(1,1)$ has an even number of bicolored edges.
- Each of the other three edges of the square contains has an odd number of bicolored edges.
- $\Rightarrow$ By Sperner's lemma the triangulation contains a tricolored triangle T.
- By the previous slide, $|\text{Area}(T)|_2' > 1$.
- But $\text{Area}(T) = \frac{1}{m}$, and $|\frac{1}{m}|_2' > 1$ only if m is even!

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An hint: Use the property that every straight line is colored by at most two colors to reduce the proof for dissections to the proof we've already shown for triangulations.

Generalizations:

Mead If the unit cube in $\mathbb{R}^n$ is dissected into m equi-volume simplices then $n!|m$.

Kasimatis Let $n \geqslant 5$ be an integer. An $n-$gon can be dissected into m triangles only if $n|m$.

Thank you for listening!
Congratulations!
And good luck!