Solutions - first stage of Israeli students competition, 2020.

1. Compute $\lim\limits_{N\to\infty} \frac{\sqrt[N]{\prod\limits_{k=1}^{N}\left(k^k \cdot k!\right)}}{(N+1)!}$

Answer. $\frac{1}{e}$.

Solution. Firstly, let us discuss $\prod\limits_{k=1}^{N}\left(k^k \cdot k!\right)$. It turns out it may be rewritten in a more simple way. Each number $j=1,2,3,...,N$ appears in the product in power j at term j; it also appears as a term in $k!$ for $k=j, j+1,...,N$ so it is multiplied totally $N+1$ times.

Hence $\prod\limits_{k=1}^{N}\left(k^k \cdot k!\right) = \left((N+1)!\right)^{N+1}$. Hence

$$\frac{\sqrt[N]{\prod\limits_{k=1}^{N}\left(k^k \cdot k!\right)}}{(N+1)!} = \frac{\sqrt[N]{(N!)^{N+1}}}{(N+1)!} = \frac{N!\cdot\sqrt[N]{N!}}{(N+1)!} = \frac{\sqrt[N]{N!}}{N+1}$$

We have simplified the expression , now we shall need a lemma

Lemma. If a sequence of positive numbers a_n converges to a positive number α, then $b_n = \sqrt[n]{a_1 a_2 \cdot ... \cdot a_n}$ converges to the same number.

To prove the lemma, one might take $\ln$ to both sequences and transform it to a lemma about arithmetic means (and real numbers instead of positive); we shall leave it as an easy exercise to the reader.

Equivalent form of this lemma is the following: for any positive sequence b_n, we may say that $\lim b_n = \lim \frac{b_n^n}{b_{n-1}^{n-1}}$, at least if $\lim \frac{b_n^n}{b_{n-1}^{n-1}}$ is defined.

Now we may finish the solution:

$$\lim \frac{\sqrt[N]{\prod_{k=1}^{N}\left(k^k \cdot k!\right)}}{(N+1)!} = \lim \frac{\sqrt[N]{N!}}{N+1} \underset{(hopefully)}{=} \lim\left(\left(\frac{\sqrt[N]{N!}}{N+1}\right)^N \bigg/ \left(\frac{\sqrt[N]{(N-1)!}}{N}\right)^{N-1}\right) =$$

$$= \lim\left(\left(\frac{\sqrt[N]{N!}}{(N+1)^N}\right)^N \bigg/ \left(\frac{\sqrt[N]{(N-1)!}}{N}\right)^{N-1}\right) = \lim\left(\frac{N!}{(N+1)^N} \bigg/ \frac{(N-1)!}{N^{N-1}}\right) =$$

$$= \lim\left(\frac{N^N}{(N+1)^N}\right) = \lim \frac{1}{\left(1+\frac{1}{N}\right)^N} = \frac{1}{e}$$

Since the limit after the trick turned out to be defined, it is equal to the original limit.

2. What is the maximal possible value of $\det\begin{pmatrix} * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \end{pmatrix}$, if each $*$ is replaced by 0 or 1?

(Different stars may be replaced by different numbers.)

Answer. 2

Solution. We may change the order of rows and the order of columns to get a matrix with $\pm$ the same determinant.

$$\det\begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

We may change the places of first and second rows to get a matrix with the same pattern. Therefore we may forget about $\pm$. The determinant is a product of upper-left 3×3 corner determinant and of 2×2lower-right corner determinant. We may assume both are positive (or else permute two rows in each).

Let us first discuss $\max\det\begin{pmatrix} * & * \\ * & * \end{pmatrix}$, where each $*$ is zero or one. If one of the rows is $0 \quad 0$ then the determinant is zero. If both rows are $1 \quad 1$ the determinant is zero as well. So we assume that one of the rows has both zero and 1. We may assume that another row has zero where this row has 1 (otherwise subtract that row from it, it doesn't affect the determinant.

If determinant is nonzero, the other row also has 1. So both rows have 1 at different columns, so the matrix is a unit matrix, maybe with different order of rows, so it is 1 at most.

Now let us discuss $\max\det\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$. If one of the rows is zero, $\det = 0$.

If one of the rows has only one 1, WLOG (up to permutations) we might say that the first row has 1 at the first column, we may subtract the first row from any othe row to ensure the first column have zeroes at all other rows, and then the determinant is equivalent to a 2×2 question and then it is 1 at most.

So, from now on assume that all rows have at least 2 ones. If one of the rows has all three ones, we may subtract another row from it. Then it have be just one 1, so we come back to the previous case. Hence we get precisely two ones in each row, but at different places each row, otherwise $\det = 0$. Then our matrix is $\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2$ up to a sign.

So $\max\det\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = 2$, and $\max\det\begin{pmatrix} * & * \\ * & * \end{pmatrix} = 1$, and the product is 2.

3. On an island, any two people who are not friends have precisely two common friends, and any two friends don't have common friends. Prove that each person has the same number of friends.

Solution. Assume some person, say A, has n friends. Each two friends of A, say B and C, have A as a common friend, and have precisely one common friend aside from A, say D, and D is not a friend of A (since friends don't have common friends). On the other hand, each D who is not a friend of A has precisely two common friends with A.

So, there is one-to-one correspondence between pairs of friends of A and non-friends of A. So there are precisely $\frac{n(n-1)}{2}$ people except A and friends of A. So, there are $\frac{n(n-1)}{2}+n+1$ people on island. This expression is a monotonically increasing function of n; hence n is the same for all people.

Remark. It is interesting to understand what is possible bumber of people on the island. Even from the above solution it follows that it can not be just any number.

One obvious example of such a graph is C_4 – a cycle of length 4.

The next example, which is already more interesting, is the following graph (16 vertices, each of degree 5): the vertices of the graph are all the ways to separate the set of five elements $\{a,b,c,d,e\}$ into 2 disjoint classes (the classes don't have names). The two separations are adjacent, if one can be obtained from another by transferring just precisely one element to the opposite class.

We suggest to the reader to find more examples or even to describe all such graphs.

4. Find the greatest $m\in\mathbb{R}$, such that $\int_0^5 f(x)dx \ge m$ for any convex function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(0)=0,\quad f(1)=1^2,\quad f(2)=2^2,\quad f(3)=3^2,\quad f(4)=4^2,\quad f(5)=5^2.$$

Answer. 40

Solution. We divide the solution into parts:

Lemma 1. For any convex function, $\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)dx \ge f(a)$. If f is linear on $\left[a-\frac{1}{2}, a+\frac{1}{2}\right]$ it is equality.

Lemma 2. If a convex function f and a linear function f satisfy: $f(x_1) = y_1 = \ell(x_1)$ and $f(x_2) = y_2 = \ell(x_2)$ then for any $x_3 \notin [x_1, x_2]$ we get $f(x_3) \le \ell(x_3)$.

We shall prove the lemmas in the end. Let us construct two linear functions satisfying

$$\ell_1(1) = 1 \qquad \ell_2(3) = 9$$
$$\ell_1(2) = 4 \qquad \ell_2(4) = 16$$

$$k_1 = \frac{4-1}{2-1} = 3 \qquad k_1 = \frac{16-9}{4-3} = 7$$

$$\ell_1(x) = 3x - 2 \qquad \ell_2(x) = 7x - 12$$

Notice that $\ell_1(2.5) = \ell_2(2.5)$. So we may define a function

$$g(x) = \begin{cases} \ell_1(x) & x \le 2.5 \\ \ell_2(x) & x \ge 2.5 \end{cases}$$

It is continuous since $\ell_1(2.5) = \ell_2(2.5)$, moreover, it is convex because the slope of ℓ_1 is less than the slope of ℓ_2. Any convex function satisfying the conditions satisfies $\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)dx \ge f(a) = g(a) = \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} g(x)dx$ for any integer $a \in [0,5]$ by lemma 1, since g is linear on $\left[a-\frac{1}{2}, a+\frac{1}{2}\right]$ for each integer a. Moreover, by lemma 2 for interval $[1,2]$ we have $f(x) \ge g(x)$ on $[0, 0.5]$, and by lemma 2 for interval $[3,4]$ we have $f(x) \ge g(x)$ on $[4.5, 5]$. Therefore

$$\int_0^5 f\,dx = \int_0^{0.5} f\,dx + \sum_{i=1}^{4} \int_{i-0.5}^{i+0.5} f\,dx + \int_{4.5}^{5} f\,dx \ge \int_0^{0.5} g\,dx + \sum_{i=1}^{4} \int_{i-0.5}^{i+0.5} g\,dx + \int_{4.5}^{5} g\,dx = \int_0^5 g\,dx$$

Hence the answer is $m = \int_0^5 g(x)dx = \int_0^{2.5} (3x-2)dx + \int_{2.5}^{5} (7x-12)dx = \ldots$

The area of a trapezoid is its altitude times the average of the bases, so

$$m = \frac{(3\cdot 0-2)+(3\cdot 2.5-2)}{2}\cdot\frac{5}{2} + \frac{(7\cdot 2.5-12)+(7\cdot 5-12)}{2}\cdot\frac{5}{2} =$$
$$= \frac{-2+5.5+5.5+23}{4}\cdot 5 = \frac{-2+11+23}{4}\cdot 5 = \frac{32}{4}\cdot 5 = 8\cdot 5 = 40$$

Strictly speaking, g does not satisfy all conditions; since

$g(0)<0$ and $g(5)<25$. But we can replace it by

$$\max\left(g\ ,\ g-1000x\ ,\ 25+1000\cdot(x-5)\right)$$

And it will have a similar integral, and if we replace 1000 by an even greater number the integral is even closer to 40.

5. A cubic curve passes through 9 different points:
$(0,1)$, $(0,2)$, $(0,5)$, $(1,1)$, $(1,3)$, $(1,5)$, $(3,7)$, $(3,8)$, $(3,y)$.
Find the value of y.
Reminder: a cubic curve is defined by an equation $p(x,y)=0$ where $p(x,y)$ is a polynomial of degree 3, i.e. $p(x,y)=\sum_{i+j\le 3} a_{i,j}x^i y^j$.

Answer. $y=-4$.

Solution. For each specific x, the polynomial $p(x,y)$ is a polynomial in y of degree 3:

$$ay^3+(b+cx)y^2+\left(d\cdot x^2+ex+f\right)\cdot y+\left(gx^3+hx^2+ix+j\right)=0$$

The sum of all three roots is $y_1+y_2+y_3=-\dfrac{b+cx}{a}$, which is a linear function in x.

In our case: for $x=0$ we get $y_1+y_2+y_3=1+2+5=8$.

In our case: for $x=1$ we get $y_1+y_2+y_3=1+3+5=9$.

So, $y_1+y_2+y_3=1+x$. So, for $x=3$ we should get

$7+8+y=y_1+y_2+y_3=8+3=11$

Hence $y=-4$.

Remark. The idea from the problem comes from the beautiful Reiss formula https://en.wikipedia.org/wiki/Reiss_relation .

6. Is it possible to mark 16 distinct points, so that at distance precisely 1 from each marked point there will be precisely 10 marked points

(a) in $\mathbb{R}^3$? (b) in $\mathbb{R}^4$?

Answers. (a) No. (b) Yes.

(a) **First solution.** Consider two marked points A and B so that $AB\neq 1$. On the unit sphere centered at A there should be 10 marked points. On the unit sphere of B there should be 10 marked points as well. Two sets of marked points other than A and B should have at least 6 points in common. The intersection of the two spheres is a circle.
Define **blue circles** to be intersections of two spheres of radius 1 centered at marked points at distance $\neq 1$ from each other. For each blue circle, the centers of those unit spheres are uniquely defined (since those should be the points at distance precisely one from all points of the circle, and there are only two such points in $\mathbb{R}^3$). There are precisely $\frac{16\cdot 5}{2}=8\cdot 5=40$ pairs of points at distance $\neq 1$, so there should be 40 blue circles; on each unit sphere centered at a marked point there should be precisely 5 blue circles. Each blue circle passes through at least 5 marked points.
So there should be 35 blue circles not contained in a unit sphere σ centered at a specific marked point. Each of these circles inetesects σ at no more than 2 points, so there should be at least 4 marked points on that blue circle which are not on the sphere. A circle is defined uniquely by each 3 points on it. But there are only 6 marked points not on σ, one can make only 20 triples of those points, so there might be no more than 20 blue circles (and definitely less than 35).

Second solution. Consider quadruples of distinct marked points (P, X, Y, Z) such that $PX = PY = PZ = 1$. It is easy to see that for any triple of points (X, Y, Z), there are at most 2 points P satisfying the condition, so the number of such quadruples does not exceed $2 \cdot \binom{16}{3}$. On the other hand, there are precisely 16 ways to choose a point P, which should have precisely 10 "neighbors", so the number is $16 \cdot \binom{10}{3}$.

So, if an example exists, then $2 \cdot \binom{16}{3} \geq 16 \cdot \binom{10}{3}$.

$$2 \cdot \frac{16 \cdot 15 \cdot 14}{3!} \geq 16 \cdot \frac{10 \cdot 9 \cdot 8}{3!}$$
$$2 \cdot 15 \cdot 14 \geq 10 \cdot 9 \cdot 8$$
$$3 \cdot 14 \geq 9 \cdot 8$$
$$7 \geq 3 \cdot 4$$

And that is wrong.

(b) Take a sphere of radius $\frac{1}{\sqrt{2}}$ centered at (0,0,0,0). Planes $\{x = y = 0\}$ and $\{z = w = 0\}$ intersect the sphere at 2 circles. On each of those two circles, mark 8 vertices of a convex octagon. Those are 16 points. For each vector we get 2 ortogonal vectors on the same circle (since 8 is divisible by 4) and 8 orthogonal vectors on the other circles, so 10 ortogonal vectors in total. Ortogonal vectors correspond to points at distance precisely 1 by Pythagorean theorem.

Remark. The example is related to a nice question I got from Shachar Carmeli and Tsachik Gelander. It is known that in any decomposition of $\mathbb{S}^2$ into convex polygonal cells there is a cell with at most 5 sides. Is there an analogy for higher dimension, say for a decomposition of $\mathbb{S}^3$ into convex polyhedral cells, can you prove that one of the cells has less than 100 faces?
The construction from presented in the solution is related to a nice counter-example.

If you take Voronoy cells for the set of marked points for the example in the (*b*) part of this problem, you have $\mathbb{S}^3$ decomposed into prisms with 8 faces each, but if we would replace 8 points on both circles by N points, Voronoy cells would be prisms with $N+2$ faces each.