First stage of Israeli students competition, 2018.
Please try to write your solutions in English.
Duration: 4 hours
את השאלון ניתן לקחת איתך בסיום התחרות.

1. (*a*) Consider an infinite set S consisting of $2^{\aleph_0}$ elements. Each two vertices of S are connected by an edge, and each edge is painted in one of $\aleph_0$ colors. Prove or disprove that there is a monochromatic triangle, i. e. a triangle with 3 sides of the same color.
(*b*1) [For those who proved the statement in *a*.] For which maximal k, you can prove that there is a complete monochromatic graph with k vertices?
(*b*2) [For those who disproved the statement in *a*.] For which maximal cardinality of S we can disprove the existence of monochromatic triangle (still with $\aleph_0$ colors)?

Solution. We shall prove there doesn't have to be a triangle. This way, we shall have to solve (*b*2) and not (*b*1).
(a) Let S be a set of infinite sequences of zeroes and ones, such as

(0010101100….)

For every two different sequences, we shall choose a color of an edge connecting them. The colors will be natural numbers. The color will be the index of the first bit at which the sequences differ.
We have $2^{\aleph_0}$ and $\aleph_0$ colors. We claim there are no monochromatic triangles. Indeed, if vertices U,V,W are connected by edges of color k then the first $k-1$ bits of all sequences are the same, and at bit number k each two are different, but there are only 2 different bits.
(b) We shall prove that there can be no greater number of vertices; that if there are more than $2^{\aleph_0}$ then there has to be a monochromatic triangle.
In other words, we shall construct an injective function from S to a set of $2^{\aleph_0}$ elements,
We shall choose a complete ordering on the set S, so that in any subset we would be able to choose a minimal element.
What we shall do next is in the spirit of the proof of the usual Ramsey theorem, but with transfinite induction. Let r be the minimal element of S, then divide $S\setminus\{r\}$ into a disjoint union of subsets S_i, where S_i will be the set of all vertices which are connected to r by an edge of color i.

Next, we choose a minimal number in each S_i and denote it r_i. Notice, that we may assume that neither two vertexes in S_i are connected by an edge of color i, otherwise we get a triangle.
Divide $S_i \setminus \{r_i\}$ into subset $S_{i,j}$ as follows: $S_{i,j}$ will be the set of all vertices in S_i, which are connected to r_i by an edge of color j.
Then we may define $r_{i,j}$ as minimal element of $S_{i,j}$, and divide it into subsets $S_{i,j,k}$ and so on. Let us give a general definition. For any complete ordered subset O of the set of colors, we shall define the subset S_O of vertexes, and then define its smallest element r_O. If O is successive ordinal, obtained by adding the extra last color ℓ to the ordinal P, we define S_O as a set of all vertexes in S_P which are connected to r_O by an edge of color ℓ. Of course, colors of edges connecting some two vertices in S_O must be different from all colors in O, because otherwise we get a complete triangle.
Let ordering on O be a limiting ordinal. Consider all subsets of T such that for any $t \in T$, and any $o \in O$ such that $o < t$, necessarily $o \in T$ (one could say that T is a beginning of O). Then we define S_O as intersections of all S_T for all such T.
So, we defined S_O and r_O for all ordinals, both successive and limiting. Some S_O can be empty, in this case r_O will be non-existent. But anyway, we won't arrive to non-countable ordinals, since the set of colors is countable. If O contains all colors, then $|S_O|$ may have at most one vertex, otherwise we get a monochromatic triangle.
So, by an ordering O of a subset of the set of colors we get a unique vertex r_O, and each vertex may be obtained thus.
The set of subsets of the set of colors is of cardinality $2^{\aleph_0}$. Each such subset has at most $\aleph_0$ elements, and at most $\aleph_0$ pairs elements. To choose an ordering, it is enough to say for each pair of elements which is greater, which gives $2^{\aleph_0}$ at most (not any decision gives an order relation, and not for each order relation we have a minimal element in each subset).
Anyway, we get at most $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$ orderings of subsets, so at most vertexes $2^{\aleph_0}$ in the graph.

2. For $x < 0$, prove that $2^x + 2^{\frac{1}{x}} \le 1$.

Solution. Notice that for $x=-1$ we get equality: $\frac{1}{2}+\frac{1}{2}=1$. Also, $x\to-\infty$ and for $x\to 0-$. However, for $x=-2$ of $x=-\frac{1}{2}$, we get $2^{-2}+2^{-\frac{1}{2}}=\frac{1}{4}+\frac{1}{\sqrt{2}}<1$, because $\frac{1}{\sqrt{2}}<\frac{3}{4}$, since $\frac{1}{2}<\frac{9}{16}$.

We want to investigate growth and decline of function for negative x. To have more symmetry, we denote $x=-e^t$, and the function we want to investigate is $f(t)=2^{-e^t}+2^{-e^{-t}}-1$.

Then $f'(t)=-(\ln 2)\cdot 2^{-e^t}\cdot e^t+(\ln 2)\cdot 2^{-e^{-t}}\cdot e^{-t}$. The question of whether f' is positive of negative is the same as whether

$$-2^{-e^t}\cdot e^t+2^{-e^{-t}}\cdot e^{-t}\overset{?}{\geq}0$$

$$2^{-e^{-t}}\cdot e^{-t}\overset{?}{\geq}2^{-e^t}\cdot e^t$$

Applying $\ln$ to both sides, we get an equivalent question, whether

$$-\ln(2)e^{-t}-t\overset{?}{\geq}-\ln(2)e^t+t$$

$$\ln(2)\left(e^t-e^{-t}\right)\overset{?}{\geq}2t$$

$$\frac{e^t-e^{-t}}{2}\overset{?}{\geq}\frac{t}{\ln(2)}$$

So it is a question about which graph is higher at a certain point: of $\frac{t}{\ln 2}$ or of $\mathrm{sh}(t)=\frac{e^t-e^{-t}}{2}$. Notice, that the function $\mathrm{sh}(t)$ is convex for $t\geq 0$ and concave for $t\leq 0$ (since its second derivative is itself).

Since all relevant functions of t are odd, it is enough to investigate the case of $t\geq 0$. Since $\mathrm{sh}(t)$ is convex for $t\geq 0$, its graph can intersect a line at most two times. One of intersections is at $t=0$. Notice $\mathrm{sh}'(0)=1$, while $\ln 2<1$, so $\mathrm{sh}(0.0001)>\frac{0.0001}{\ln 2}$. On the other hand, for large t,

say 1000, we have $\mathrm{sh}(1000) > \frac{1000}{\ln 2}$. By continuity, there exists $t_1 > 0$ inside $[0.0001, 1000]$ such that $\mathrm{sh}(t_1) = \frac{t_1}{\ln 2}$.

Therefore for $t \in (0, t_1)$ the derivative is negative, and for $t \in (t_1, \infty)$ the derivative is positive. So f is decreasing on $[0, t_1]$ and is increasing on $[t_1, \infty)$.

So each value of f for positive t is less than $f(0)$ or than $\lim_{t\to\infty} f(t)$, which are both 1. So $f(t)$ is not positive for positive t, and the same is true for negative t since f is even.

3. Find all points $P = (r, 0)$ on the horizontal axis, $r \in \mathbb{Q}$, such that the distances from P to the vertices of the square $(\pm 1, \pm 1)$ are all rational?

Solution. We know that $(r-1)^2 + 1 = w_1^2$ and $(r+1)^2 + 1 = w_2^2$, where $w_1, w_2 \in \mathbb{Q}$. In other words $r^2 - 2r + 2 = w_1^2$ and $r^2 + 2r + 2 = w_2^2$.

The product of those two formulas gives $r^4 + 4 = w^2$, where $w = w_1 w_2$. Multiplying by the denominator, we get $x^4 + 4y^4 = z^2$.

Definition. A natural number will be called ***squarish***, if it is of the form n^2 or of the form $2n^2$, where n is a natural number.

Lemma. If (a, b, c) is a Pythagorean triple (such that $a^2 + b^2 = c^2$) and at least two numbers in the triple are squarish, then either a or b is zero.

From the lemma we may conclude that in our case $(x^2, 2y^2, z)$ either x or y is zero. So in $r^4 + 4 = w^2$ we get $r = 0$.

So the only point to consider is the origin, but the distance from the origin to the vertices of the square are $\sqrt{2}$ and definitely not rational (even if their product is rational.

Proof of the lemma. Suppose we have $a^2+b^2=c^2$, and neither among a,b,c is zero, and two of them are squarish. We may take the example with minimal $|c|$. We claim that the numbers are pairwise co-prime, since if p divides two of them, it divides all of them, and p^4 divides two terms of the equation, so also the third (except for the case $p=2$). By dividing a,b and c by p^2 (or by 2) we get a smaller example.

By Pythagorean triples classification theorem, we have $c=n^2+m^2$ and $b=2mn$, $a=m^2-n^2$, where b is even and a,c are odd.

Then m,n are co-prime. If b is squarish then m,n are both squarish and either a or c are squarish. But a,c are odd, so one of them is a square, so we get $d^2=m^2\pm n^2$ where d is $\sqrt{c}$ or $\sqrt{a}$, either way we have a smaller Pythagorean triple with 2 squarish numbers.

If a,c are both squarish then $ac=m^4-n^4$, but a,c are squares so we get again a smaller Pythagorean triple.

4. We are given a sphere and a tetrahedron, such that each edge of the tetrahedron is tangent to the sphere at 6 different points, and the center of the sphere is strictly inside the tetrahedron. Show that there exists another tetrahedron, also having all edges tangent to the sphere, with precisely the same points of tangency as for the given tetrahedron.

Solution. Let A be a point outside a sphere. Then the lines from A form tangent to the sphere can be moved one into another, by rotation around the line connecting A to the center of the sphere. Hence all points of tangency of sphere with lines passing through A and tangent to the sphere, are in one plane.

Let ABC be a face of the tetrahedron. Its edges are tangent to the sphere at points U, V, W. The face intersects the sphere along a circle, and U, V, W are on this circle. The circle can be obtained as points of tangency from some point P outside the sphere. The lines PU, PV, PW are perpendicular to respective edges.

Therefore, if at each point of tangency we take a line perpendicular to a respective edge, we get 6 lines, and lines obtained from the same face meet at one point, so we get another tetrahedron.

5. Let A be an $n \times n$ anti-symmetric matrix (meaning $\mathrm{A} = \mathrm{A}^{\mathrm{T}}$) with real entries, its columns are $A_1, A_2, ..., A_n$. Show that there exist n nonnegative numbers $x_1, x_2, ..., x_n$ not all of them zero, such that all the coordinates of the vector $x_1 A_1 + x_2 A_2 + ... + x_n A_n$ are nonnegative.

First solution. Let $e_1, e_2, ..., e_n$ denote the vectors of the standard basis. A ***cone*** spanned by $v_1, ..., v_k$ means $\{\alpha_1 v_1 + ... + \alpha_k v_k \mid \alpha_1, ..., \alpha_k \geq 0\}$. If the cone spanned by $e_1, e_2, ..., e_n$ doesn't have a common internal point with the cone spanned by $A_1, A_2, ..., A_n$, by Hahn-Banach theorem, there is a linear functional, which can be written as a row vector ℓ, such that $\ell A_i \leq 0$, but $\ell e_i \geq 0$; not all coordinates of ℓ are zero, but all are nonnegative. Also, $\ell \mathrm{A}$ has only non-positive coordinates. Transpose this inequality, and revert the sign: $\mathrm{A} \cdot \ell^{\mathrm{T}}$ has only nonnegative coordinates. But $\mathrm{A} \cdot \ell^{\mathrm{T}}$ is a linear combination of $A_1, A_2, ..., A_n$ with nonnegative coefficients, Q.E.D.

Second solution. Let S be a set of all vectors in $\mathbb{R}^n$, having nonnegative coordinates with a total sum 1. For any $v \in S$, the vector Av has at least one negative coordinate, otherwise we are done.

Construct a function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} x, & x \leq 0 \\ 0, & x = 0 \end{cases}$, in other words making numbers non-positive, and $F : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

(making all coordinates non-positive). F has some nice properties:

(1) For any vector $v \in \mathbb{R}^n$, we get $\langle v, F(v) \rangle \geq 0$.

(2) For any vector $u \in \mathbb{R}^n$ having a negative coordinate, $F(u) \neq 0$.

Therefore, if $v \in S$, $F(Av) \neq 0$.

For $v \in S$, the vector $w = F(Av) = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ has no positive coordinates, and some coordinates are strictly negative, so the vector $z = \frac{w}{w_1 + ... + w_n}$ has the opposite direction to w, and satisfies $z \in S$. Thus we have constructed a continuous mapping

$$S \to S$$
$$v \mapsto z$$

By Brouwer fixed-point theorem, there exists $v \in S$, such that $z = v$, which means that a direction of w is strictly opposite to the direction of v. Hence $\langle \mathrm{A}v, w \rangle > 0$, so $\langle \mathrm{A}v, v \rangle < 0$, so $v^{\mathrm{T}} \mathrm{A} v < 0$.

But $v^{\mathrm{T}} \mathrm{A} v = 0$, since matrix A ix anti-symmetric.

6. Consider a sequence $\{a_n\}$ of real numbers. Denote $s_k = \sum_{k=1}^{n} a_k$. It is given that $\frac{1}{n} \sum_{k=1}^{n} s_k \to \alpha \in \mathbb{R}$.

(*a*) Assuming $a_n = o\left(\frac{1}{n}\right)$. Show that $s_k \to \alpha$.

(*b*) Assuming $a_n = O\left(\frac{1}{n}\right)$. Show that $s_k \to \alpha$.

Solution. For convenience, we denote $M_n = \frac{1}{n} \sum_{k=1}^{n} s_k$.

(a) The numbers M_n are bounded, since any convergent sequence is bounded. Hence $|M_n| \leq M$. For any $\varepsilon > 0$ there is N such that for any $n > N$ we have $|a_n| < \frac{\varepsilon}{n}$. So, if we take $\varepsilon = \frac{1}{k^2}$ then

$$|a_n|, |a_{n+1}|, ... \leq \frac{1}{k^2 n}$$

Therefore, if $n \le m \le kn$ we get $\left|s_m - s_{kn}\right| \le \frac{1}{k}$ (*).

Also, we may see that

$$M_{kn} = \frac{n}{kn} M_n + \frac{kn-k}{kn}\left(s_{n+1} + ... + s_{kn}\right) = \frac{1}{k} M_n + \frac{n-1}{n}\left(s_{n+1} + ... + s_{kn}\right)$$

So

$$\left|M_{kn} - \frac{k-1}{k} \cdot s_{kn}\right| = \left|\frac{n}{kn} M_n + \frac{1}{kn}\left(\left(s_{n+1} + ... + s_{kn}\right)\right) - \frac{kn-n}{kn} s_{kn}\right| \le$$

$$\le \frac{M}{k} + \left|\frac{1}{kn} \cdot \sum_{m=n+1}^{kn} \left(s_m - s_{kn}\right)\right| \le \frac{M}{k} + \left|\frac{1}{kn} \cdot \frac{1}{k}\left(k-1\right)n\right| \le \frac{M+1}{k}$$

For n large enough, M_{kn} is as close to α as you wish, so if we chose k large enough, $\frac{k-1}{k} \cdot s_{kn}$ is as close to α as you wish, so s_{kn} is as close to $\frac{k\alpha}{k-1}$ as you wish, so s_{kn} is sufficiently close to α. And by (*) also any s_m for $m > n$ is as close to α as you wish.

(*b*) We shall reduce this part to the previous part. We shall organize a process, in which the sequence is transformed while convergence of s_k and of M_k won't be affected, and the sequence a_k will become $o(n)$ instead of $O(n)$.

The main step in the process will be inserting zeroes. By inserting a zero at place n we mean replacing the sequence a_n by the sequence

$$\tilde{a}_1 = a_1, \tilde{a}_2 = a_2, ..., \tilde{a}_{n-1} = a_{n-1}, \tilde{a}_n = 0, \tilde{a}_{n+1} = a_n, \tilde{a}_{n+2} = a_{n+1}, \tilde{a}_{n+3} = a_{n+2}, ...$$

In this step the property $s_k \to \alpha$ is preserved.

Now, we shall verify, what happens to $\lim\left(\frac{1}{n}\sum_{k=1}^{n} s_k\right)$ is we insert zeroes at places $p, 2p, 3p, 4p, ...$ (one should imagine a large p, say 999999).

Denote $q = p - 1$. For $\frac{1}{n}\sum_{k=1}^{n} s_k$, where n is large, we get a weighted average of former $\frac{1}{m}\sum_{k=1}^{m} s_k$ (where $m \approx \frac{p-1}{p} n$ and weight is $\approx p-1$) and

of $\frac{1}{\ell}\sum_{k=1}^{\ell} s_{kp}$ (where $m \approx \frac{n}{p}$ and weight is ≈ 1). Notice, that both averaged numbers $\to \alpha$. Indeed, it was given that $\frac{1}{m}\sum_{k=1}^{m} s_k \to \alpha$.

Regarding the second number: $\frac{1}{\ell}\sum_{k=1}^{\ell} s_{kp}$. For large ℓ, it can be separated into two summands: $\frac{1}{\ell}\sum_{k=1}^{L} s_{kp} + \frac{1}{\ell}\sum_{k=L+1}^{L} s_{kp}$. The first summand tends to zero, the second is sufficiently close $\frac{1}{\ell p}\sum_{k=Lp+1}^{\ell p} s_k$, since for $|kp-n| < p$ differences are small $|s_{kp} - s_n| < \frac{Cp}{L^2}$ for $kp, n > L$ and $\frac{1}{\ell p}\sum_{k=Lp+1}^{\ell p} s_k$ is sufficiently close to α.

Inserting zeroes at places $p, 2p, 3p, 4p, \ldots$ will be called p-diluting.

We have explained, that after p-diluting, both $\frac{1}{n}\sum_{k=1}^{n} s_k$ and s_k have the same limit. After p-diluting p times consequently, the density of the original sequence in the new sequence is $\left(1-\frac{1}{p}\right)^p$.

Let $p_n = 10^{10 \cdot n}$. We shall p_1-dilute p_1 times, then p_2-dilute p_2 times, then p_3-dilute p_3 times, and so on.

The sequence until $N > p_{k+1}$ became approximately at least e^k less dense. So the bound $|a_n| < \frac{C}{n}$ becomes $|a_n| < \frac{C}{e^k n}$, where k is the maximal numbers such that $N > p_{k+1}$. But $k \underset{n\to\infty}{\to} \infty$, so we get $a_n = o(n)$.

On the other hand, no element of the original sequence is shifted more that finite number of times, so $\lim s_k$ is still the same as in the beginning.