Second stage of Israeli students competition, 2018.
Please try to write your solutions in English.
Duration: 4 hours
את השאלון ניתן לקחת איתך בסיום התחרות.

1. (*a*) Consider an infinite set S consisting of $2^{\aleph_0}$ elements. Each two vertices of S are connected by an edge, and each edge is painted in one of $\aleph_0$ colors. Prove or disprove that there has to be a monochromatic triangle, i. e. a triangle with 3 sides of the same color.
(*b*1) [For those who proved the statement in *a*.] For which maximal k, you can prove that there is a complete monochromatic subgraph with k vertices?
(*b*2) [For those who disproved the statement in *a*.] For which maximal cardinality of S we can disprove the existence of monochromatic triangle (still with $\aleph_0$ colors)?

2. For $x<0$, prove that $2^x+2^{\frac{1}{x}}\le 1$.

3. Find all points $P=(r,0)$ on the horizontal axis, $r\in\mathbb{Q}$, such that the distances from P to the vertices of the square $(\pm 1,\pm 1)$ are all rational?

4. We are given a sphere and a tetrahedron, such that each edge of the tetrahedron is tangent to the sphere at 6 different points, and the center of the sphere is strictly inside the tetrahedron. Show that there exists another tetrahedron, also having all edges tangent to the sphere, with precisely the same points of tangency as for the given tetrahedron.

5. Let A be an $n\times n$ anti-symmetric matrix (meaning $\mathrm{A}=-\mathrm{A}^{\mathrm{T}}$) with real entries, its columns are $A_1,A_2,...,A_n$. Show that there exist n nonnegative numbers $x_1,x_2,...,x_n$ not all of them zero, such that all the coordinates of the vector $x_1A_1+x_2A_2+...+x_nA_n$ are nonnegative.

6. Consider a sequence $\{a_n\}$ of real numbers. Denote $s_n=\sum_{k=1}^{n}a_k$. It is given that $\frac{1}{n}\sum_{k=1}^{n}s_k\to\alpha\in\mathbb{R}$.
(*a*) Assuming $a_n=o\left(\frac{1}{n}\right)$, show that $s_k\to\alpha$.
(*b*) Assuming $a_n=O\left(\frac{1}{n}\right)$, show that $s_k\to\alpha$.

Good luck!