First stage of Israeli students competition, 2018.

Please try to write your solutions in English.

1. Compute $\det\begin{pmatrix}1&2&2&2&2\\1&3&4&4&4\\1&3&5&6&6\\1&3&5&7&8\\1&3&5&7&9\end{pmatrix}$.

Answer. 1.

First solution. Subtract 4th row from the last, then subtract 3rd row from the 4th, then subtract 2nd row from the third, then 1st row from the 2nd.

$$\det\begin{pmatrix}1&2&2&2&2\\1&3&4&4&4\\1&3&5&6&6\\1&3&5&7&8\\1&3&5&7&9\end{pmatrix}=\det\begin{pmatrix}1&2&2&2&2\\1&3&4&4&4\\1&3&5&6&6\\1&3&5&7&8\\0&0&0&0&1\end{pmatrix}=\det\begin{pmatrix}1&2&2&2&2\\1&3&4&4&4\\0&0&1&2&2\\0&0&0&1&2\\0&0&0&0&1\end{pmatrix}=$$

$$=\det\begin{pmatrix}1&2&2&2&2\\0&1&2&2&2\\0&0&1&2&2\\0&0&0&1&2\\0&0&0&0&1\end{pmatrix}=1$$

In the end we get an upper-triangular, matrix, so the determinant is the product of the diagonal.

Second solution. Let us solve a more general question:

$$\det\begin{pmatrix}a&b&b&b&b\\a&c&d&d&d\\a&c&e&f&f\\a&c&e&g&h\\a&c&e&g&i\end{pmatrix}$$

Where $a,b,c,...,i$ are arbitrary numbers. Notice that if $a=0$ the matrix is degenerate. If $c=b$, the first two columns are linearly dependent, so matrix is linearly dependent.

If $d=e$, among three left columns all lines except the firs are the same, so the rank of the matrix consisting of 3 first columns is 2, so the first three columns are linearly dependent. So the matrix is degenerate.

For similar reasons, matrix is degenerate if $f=g$ or if $h=i$.
Determinant is a polynomial in $a,b,c,...,i$.
It must be divisible by a, by $c-b$, by $e-d$, by $g-f$, and by $i-h$.
So it is divisible $a(c-b)(e-d)(g-f)(i-h)$. It is a polynomial of degree 5, so it is $a(c-b)(e-d)(g-f)(i-h)$ multiplied by a constant.
To understand, what is the constant substitute $b=d=f=h=0$. We get a lower-triangular matrix, the determinant is a product of diagonal elements $a\cdot c\cdot e\cdot g\cdot i$, so the determinant is

$$a(c-b)(e-d)(g-f)(i-h)$$

Now substitute $a=1, b=2, c=3, \dots$, we get the answer.

2. Consider the polynomial $p(x)=(x+1)^{90}+(x-1)^{90}$.

(a) Find all complex roots of p.

(b) Compute $\tan^2(1^\circ)+\tan^2(3^\circ)+\tan^2(5^\circ)+\tan^2(7^\circ)+...+\tan^2(89^\circ)$.

Answer. $\binom{90}{2}=4005$.

Solution. In order for x to be a root, the numbers $(x+1)^{90}$ and $(x-1)^{90}$ have to have the same absolute value but opposite arguments.
If they have the same absolute values, the distance from x to 1 is the same as the distance from x to -1. Hence x is on the perpendicular bisector of 1 and -1, so x is imaginary. So we shall right $x=i\cdot t$.
Notice that $t=\tan\alpha$, where α is the argument of $t+1$. The argument of $t-1$ is $180^\circ-\alpha$. Hence the arguments of $(x+1)^{90}$ and $(x-1)^{90}$ are $90\cdot\alpha$ and $-90\cdot\alpha$, so $90\cdot\alpha=180^\circ-90\cdot\alpha\left(\mathrm{mod}\,360^\circ\right)$.
In other words, $180\cdot\alpha=180^\circ+360^\circ k$, where k is in integer number, therefore $\alpha=(2k+1)^\circ$.
So, possible roots are $i\cdot\tan1^\circ, i\cdot\tan3^\circ, i\cdot\tan5^\circ, i\cdot\tan7^\circ, ..., i\cdot\tan179^\circ$.
These are 90 distinct roots can be divided into $\pm$ pairs, since

$$\tan179^\circ=-\tan1^\circ,\ \tan177^\circ=-\tan3^\circ,\ \tan175^\circ=-\tan5^\circ,...$$

There are no other roots, both by our previous computations and since a polynomial of degree 90 can have no more than 90 roots. Notice that $(x+i\cdot t)(x-i\cdot t)=x^2+t^2$, so in our case we get a polynomial in $y=x^2$ which has 45 roots, which are precisely $-\tan^2 1^\circ, -\tan^2 3^\circ, ..., -\tan^2 89^\circ$, so we just need to compute sum of roots of polynomial in y.

In our case, we shall divide the polynomial by 2, it doesn't influence the roots, and our polynomial becomes

$$x^{90}+\binom{90}{2}\cdot x^{88}+\ldots=y^{45}+\binom{90}{2}\cdot y^{44}+\ldots$$

So sum of roots by Vieta is

$$-\left(\tan^2 1^\circ+\tan^2 3^\circ+\tan^2 5^\circ+\tan^2 7^\circ+\ldots+\tan^2 89\right)=-\binom{90}{2}$$

$$\tan^2 1^\circ+\tan^2 3^\circ+\tan^2 5^\circ+\tan^2 7^\circ+\ldots+\tan^2 89=\binom{90}{2}=45\cdot 89=4005$$

3. Prove that $\lim\limits_{t\to 1-}\left(\sqrt[3]{1-t}\cdot\left(1+t+t^8+t^{27}+\ldots+t^{n^3}+\ldots\right)\right)=\int\limits_0^\infty e^{-x^3}dx$.

Solution. We shall try to estimate the integral. Let us divide the positive ray into small intervals of length δ. The value at node number n is $e^{-\delta^3\cdot n^3}$, so if we denote $e^{-\delta^3}=t$.

Therefore, the approximation of the integral is $\delta\cdot\sum\limits_{n=0}^{\infty}t^{n^3}$.

As δ tends to 0, the value of this approximation tends to the integral, and t tends to 1 from below. Let us express it via t. If $e^{-\delta^3}=t$, then $-\delta^3=\ln t$, and $-\delta^3=\ln t$, so $\delta=-\sqrt[3]{\ln t}$. Notice that $\lim\limits_{t\to 1}\dfrac{\ln t}{t-1}=1$.

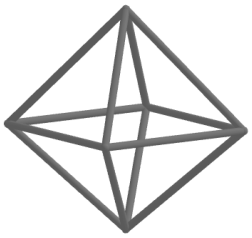
Therefore $\lim\limits_{t\to 1}\dfrac{-\sqrt[3]{\ln t}}{\sqrt[3]{1-t}}=1$. Hence

$$\int\limits_0^\infty e^{-x^3}dx=\lim_{t\to 1-}\left(-\sqrt[3]{\ln t}\cdot\left(1+t+t^8+t^{27}+\ldots+t^{n^3}+\ldots\right)\right)=$$

$$=\lim_{t\to 1}\frac{-\sqrt[3]{\ln t}}{\sqrt[3]{1-t}}\cdot\lim_{t\to 1-}\left(\sqrt[3]{1-t}\cdot\left(1+t+t^8+t^{27}+\ldots+t^{n^3}+\ldots\right)\right)=$$

$$=\lim_{t\to 1-}\left(\sqrt[3]{1-t}\cdot\left(1+t+t^8+t^{27}+\ldots+t^{n^3}+\ldots\right)\right).$$

4. A wolf hunts a rabbit on a space station, consisting of 12 corridors which form a regular octahedron. The wolf can see the rabbit only if they are in the same corridor. Assuming the velocity of the wolf is 2.5 times greater than the velocity of the rabbit. Can the



wolf devise a strategy to catch the rabbit in any case?

Answer. Yes.

Solution. Denote the vertexes of the octahedron by A, B, C, D, E, F, where A is opposite to D, B is opposite to E, C is opposite to F.

Wolf may run along the route (we shall call it ***the path of the wolf***)

A-B-C-D-E-F-A-B-C-D-E-F-A-B-C-D-E-F-A-B-C-D-E-F-…

until he will see the rabbit. The moment he will see the rabbit, he will run straight towards the rabbit, until he catches him.
Let us prove the wolf will see the rabbit eventually.

Consider a circle, on which 6 points are marked at equal distances.

We shall denote points on the circle A', B', C', D', E', F' in this order. For any point on one of the edges of the octahedron, we define a corresponding point on a circle, its ***shadow***, in the following way:

A shadow of A is A', a shadow of B is B', … . As a point moves with constant velocity from a vertex X to the vertex Y of the octahedron, its shadow moves with constant velocity between the shadows of X and Y, along the shortest arc. This is uniquely defined, since opposite vertices of the hexagon correspond to the opposite vertices of the octahedron.

Notice, that if an animal runs along ***the path of the wolf***, its shadow moves twice more slow then if it runs along one of the 6 other corridors. But anyway, a wolf runs 2.5 times quicker, so the shadow of the wolf will catch the shadow of the rabbit. If the rabbit is on ***the path of the wolf*** at that moment, it is caught already. If not, a rabbit is in the corridor connecting vertices X and Y, while the wolf runs from X to Y via Z, where XYZ is a face of the octahedron.
There are two options: either rabbit might be closer to X, or to Y.
In the first case, the wolf saw the rabbit when he was at point X, since the time it took for the wolf to arrive to this situation is less than it might take the rabbit to arrive to the current location from a point outside the edge XY (since it is half the distance).
In the second case, the wolf will see the rabbit when he arrives to Y, since the distance the rabbit has to pass to get off the edge XY is half the distance the wolf has to pass till it arrives to Y.
Anyway, the wolf will find the rabbit and catch it.

5. A two-dimensional square with side of length 10 is contained in a unit cube of dimension n. What is the least possible n?

Solution. The diameter of a set is the longest distance between two points in that set. The diameter of the square with side 10 is $10\sqrt{2}$, the diameter of the unit cube of dimension n is $\sqrt{n}$. So, $200 \le n$.
Now we want to insert the square into the 200-dimensional cube, and it is clear that the opposite corners of the square have to become the opposite corners of the cube, so the square and the cube have the same center, and hence we have to find 2 different corners A and B, such that the vectors OA and OB are orthogonal. Then the convex hull of A, B, $-\mathrm{A}, -\mathrm{B}$ is precisely the square we look for.
We shall choose a coordinate system such that the cube is $-\frac{1}{2} \le x_i \le \frac{1}{2}$, and take $\mathrm{A} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$, $\mathrm{B} = \left(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\right)$, so that precisely half of the coordinates of B are positive.