**Solutions - second stage of Israeli students competition, 2017.**

**1.** A permutation is a bijective function $\sigma:\{1,2,...,n\}\to\{1,2,...,n\}$.

A permutation is called even if $\prod_{i<j}(\sigma(j)-\sigma(i))>0$ and odd otherwise.

For any permutation $\sigma$ define its displacement $D(\sigma)=\prod_{i=1}^{n}|i-\sigma(i)|$.

Which is greater: sum of displacement of all even permutations or sum of displacements of all odd permutations? The answer might depend on $n$.

**Answer.** For $n$ even, odd permutation give greater total disparity, for $n$ odd, even permutations give greater total disparity. The difference is always $(n-1)2^{n-1}$.

**Solution.** Construct a $n\times n$ matrix $A=(a_{i,j})$, where $i$ is row and $j$ is column, such that $a_{i,j}=|i-j|$. The determinant of this matrix is the sum over all permutations of displacement, with sign plus for even permutation and sign minus for odd permutation. So the question is about the sign of the determinant of this matrix.

To compute the determinant of this matrix we subtract row $n-1$ from row $n$ , row $n-2$ from row $n-1$, …, row $2$ from row 3, row 1 from row 2. For example, we replace matrix

$$A=\begin{pmatrix}0&1&2&3\\1&0&1&2\\2&1&0&1\\3&2&1&0\end{pmatrix}\text{ with }B=\begin{pmatrix}0&1&2&3\\1&-1&-1&-1\\1&1&-1&-1\\1&1&1&-1\end{pmatrix}$$

with the same determinant. The first row remains the same, but the rest of the matrix turns into $\pm1$'s, ones below the main diagonal and $-1$'s elsewhere. Now we do the same for columns: we subtract column $n-1$ from column $n$, then column $n-2$ from column $n-1$, . . . , column 3 from column 2, column 2 from column 1. We get 0 at the top left entry, 1 at all other entries, and $-2$ at all entries of the main diagonal except the first. We get a matrix $C$ of the form $C=\begin{pmatrix}0&1&1&1\\1&-2&0&0\\1&0&-2&0\\1&0&0&-2\end{pmatrix}$ (of course, that is an example $4\times 4$, but our conclusions are general). Now we add $\frac{1}{2}$ of rows 2, 3, 4, … , $n-1$ to the first row. We still have a matrix with the same determinant, but now it is lower-triangular, so its determinant is easy to compute as: product of its diagonal elements. So the determinant is $\frac{n-1}{2}\cdot(-2)^{n-1}$.

**2.** Prove that $\frac{(n+2)^{n+1}}{(n+1)^n}-\frac{(n+1)^n}{n^{n-1}}<e$, for every positive integer $n$.

**Solution.** Consider function $f(x)=\frac{(x+1)^x}{x^{x-1}}$ (which is well-defined for positive $x$) .

The plan of the proof is the following.

We shall prove that $f$ is convex. The expression $f(x+1)-f(x)$ is the average slope in the interval $[x,x+1]$, so it is grows constantly. We shall show that it tends to $e$, so for any specific $x$ it is less then $e$ .

The logarithmic derivative of $f$ is

$$\frac{f'(x)}{f(x)}=\left(\ln\left(f(x)\right)\right)'=\left(x\ln(x+1)-(x-1)\ln x\right)'=$$
$$=\ln(x+1)+\frac{x}{x+1}-\ln x-\frac{x-1}{x}=\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}$$

In other words, $f'(x)=f(x)\cdot\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)$.

It allows to compute the second derivative:

$$f''(x)=\left(f'(x)\right)'=\left(f(x)\cdot\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)\right)'=$$
$$=f(x)\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)^2+f(x)\cdot\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)'=$$
$$=f(x)\left(\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)^2+\frac{1}{x+1}-\frac{1}{x}+\frac{1}{(x+1)^2}-\frac{1}{x^2}\right)$$

We want to say that this expression is nonnegative, and so the function is convex. Obviously $f(x)>0$, so it is enough to consider the bracket.

$$B=\left(\ln\frac{x+1}{x}-\frac{1}{x+1}+\frac{1}{x}\right)^2+\frac{1}{x+1}-\frac{1}{x}+\frac{1}{(x+1)^2}-\frac{1}{x^2}=$$

$$=\left(\ln\left(1+\tfrac{1}{x}\right)+\frac{1}{x(x+1)}\right)^2-\frac{1}{x(x+1)}+\frac{1}{(x+1)^2}-\frac{1}{x^2}=$$

$$=\left(\ln\left(1+\tfrac{1}{x}\right)+\frac{1}{x(x+1)}\right)^2-\frac{1}{x(x+1)^2}-\frac{1}{x^2}$$

Notice, that $y=\dfrac{1}{x}$ is convex, so $\ln(x+1)-\ln(x)=\int\limits_x^{x+1}\dfrac{dt}{t}\geq\dfrac{1}{x+\frac{1}{2}}$ (since the convex function is above tangent line). So

$$B=\left(\ln\left(1+\tfrac{1}{x}\right)+\frac{1}{x(x+1)}\right)^2-\frac{1}{x(x+1)^2}-\frac{1}{x^2}\geq$$

$$\geq\left(\frac{1}{x+\frac{1}{2}}+\frac{1}{x(x+1)}\right)^2-\frac{1}{x(x+1)^2}-\frac{1}{x^2}=$$

$$=\frac{1}{\left(x+\frac{1}{2}\right)^2}+2\cdot\frac{1}{x(x+1)\left(x+\frac{1}{2}\right)}+\frac{1}{x^2(x+1)^2}-\frac{1}{x(x+1)^2}-\frac{1}{x^2}=$$

$$=\frac{x^2-\left(x+\frac{1}{2}\right)^2}{\left(x+\frac{1}{2}\right)^2x^2}+\frac{2(x+1)-x}{x(x+1)^2\left(x+\frac{1}{2}\right)}+\frac{1}{x^2(x+1)^2}=$$

$$=\frac{-x-\frac{1}{4}}{\left(x+\frac{1}{2}\right)^2x^2}+\frac{x+2}{x(x+1)^2\left(x+\frac{1}{2}\right)}+\frac{1}{x^2(x+1)^2}=$$

$$=\frac{(x+2)x\left(x+\frac{1}{2}\right)-\left(x+\frac{1}{4}\right)(x+1)^2+\left(x+\frac{1}{2}\right)^2}{x^2(x+1)^2\left(x+\frac{1}{2}\right)^2}$$

The denominator is positive, the nominator is

$$(x+2)x\left(x+\tfrac{1}{2}\right)-\left(x+\tfrac{1}{4}\right)(x+1)^2+\left(x+\tfrac{1}{2}\right)^2=$$

$$=x^3+\frac{5}{2}x^2+x-\left(x^3+\frac{9}{2}x^2+\frac{3}{2}x+\frac{1}{4}\right)+x^2+x+\frac{1}{4}=\frac{x}{2}\geq 0$$

$$\frac{(n+2)^{n+1}}{(n+1)^n}-\frac{(n+1)^n}{n^{n-1}}=\left(1+\tfrac{1}{n}\right)^n\left(\frac{(n+2)^{n+1}}{(n+1)^n}\Bigg/\frac{(n+1)^n}{n^n}-n\right)$$

For large $n$, the first term tends to $e$, and the second term is

$$(n+2)\left(\frac{(n+2)n}{(n+1)^2}\right)^n - n = (n+2)\left(1-\frac{1}{(n+1)^2}\right)^n - n =$$

$$=(n+2)\left(1-\frac{1}{(n+1)^2}\right)^n - n = (n+2)\left(1-\frac{n}{(n+1)^2}+o\left(\tfrac{1}{n}\right)\right) - n =$$

$$= n+2-\frac{(n+2)n}{(n+1)^2}+o(1)-n = n+2-1-n+o(1) \underset{n\to\infty}{\to} 1$$

(our approximation is based on Lagrange remainder, which in special case of Newton's binomial means that $(1+x)^n = 1+nx+\binom{n}{2}y^2$, where $y$ is between 0 and $x$). So, the limit is $e$, and the slope of convex funcation is monotonically increasing, Q. E. D.

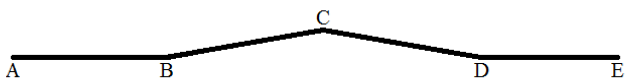
**3.** Is it possible to find a broken line (not necessary closed) inside the unit square in plane, such that its length is 10000, and each triangle with vertices on the line has

(a) an angle which is less than $15^\circ$?

(b) two angles which are less than $15^\circ$?

Answers. (a) Yes (b) No, not even line of length 2.

**Solution.** (a) We shall start with one interval of unit length, placed in the middle of a diagonal of a given square. At each stage, we shall replace an interval, say AE, by a broken line of 4 equal intervals ABCDE, where points B and D are on AE, and $\sphericalangle\mathrm{CBD} = 2^\circ = \sphericalangle\mathrm{CDB}$.




Then it is easy to see that $\sphericalangle\mathrm{CAB} = 1^\circ = \sphericalangle\mathrm{CED}$. We shall say that the rhombus (מעוין) ACEZ is the area **controlled** by the segment AE.

The point is, that the rhombi controlled by AB, BC, CD, DE are inside the rhombus controlled by AE. So if we repeat the substitution again and again, all lines are inside the rhombus controlled by AE.

Consider 3 points on the broken line, X, Y and Z, in this order. They are not in all on the same interval, their come from different quarters of some interval. We may assume that at some moment there was an interval AE which was broken, X went to one part and Y, Z went to another part or parts (there is also the case that Z is one part, X and Y on the other, but it is symmetric to our case). If X is in the first quarter AB, consider the two straight lines passing through B forming angle $3^\circ$ with AB. These lines divide the plane in 4 parts, two of which are sharp angles. The rhombi controlled by BC, CD, DE are in one sharp angle, the rhombus controlled by AD is in another, so $\sphericalangle YXZ \le 6^\circ$.

If X comes from BC (or rather, its rhombus) we construct similar two lines via C, forming angle 5 degrees with BC. In this case we see that $\sphericalangle YXZ \le 10^\circ$ (well, one of the lines can be rotated closer to BC, and the estimate can be improved, but who cares).

The case that X comes from CD, and then Y and Z come from DE, is similar to the first case.

Now, each step increases the length of the line $\alpha$ times, where $\alpha$ is greater than 1, so for some $n$ the number $\alpha^n$ will be as great as we want, which means that after $n$ iterations the broken line will be as long as we want.

**Remark.** The line we've constructed is a version of a famous curve called ***Koch snowflake***.

**(b)** Let A be an endpoint of the broken line. Let us walk from A along the line with unit velocity. We claim that we shall all the time increase the distance from A. Indeed, if at some point V we shall start getting closer to A, then there are points on the broken line, B and C, one before V and one after V, at distance $AV - \varepsilon$, where $\varepsilon$ is a very small number, and points B and C are very close to V. In this case, the triangle ABC is isosceles, with a very small $\sphericalangle A$, so the other two angles are almost $90^\circ$.

The velocity vector of our motion at point X can be decomposed into two orthogonal components, the radial and the tangent. The radial component is along the line AX and is directed outwards. The tangent component is orthogonal to AX. Consider the point Y which comes very close after X in our motion. The vector $\overrightarrow{XY}$ is in the same direction as velocity vector. The angle $\sphericalangle YAX$ is very small, the angle $\sphericalangle AXY$ is obtuse (קהה), so the angle $\sphericalangle \mathrm{AYX}$ should be also less than $15^\circ$. Actually, $\sphericalangle \mathrm{AYX}$ is very close to the angle between $\overrightarrow{\mathrm{XY}}$ and $\overrightarrow{\mathrm{AX}}$, therefore the angle between $\overrightarrow{\mathrm{AX}}$ and velocity is not greater than $15^\circ$. So the larger part of our motion is radial and not tangential. Therefore $|\mathrm{AX}|$ is increased at velocity greater than $\frac{1}{\sqrt{2}}$ at least, if we move with unit velocity. So when we pass length 2 of the curve, we will get $|\mathrm{AX}| > \sqrt{2}$ so we cannot stay inside the unit square.

**4.** Let $\alpha$ and $\beta$ be positive numbers. We construct a symmetric matrix, such that at column $i$ row $j$ the entry is $\frac{1}{i+j+\alpha} \cdot \frac{1}{i+j+\beta}$. Show that this matrix is positive definite.

**Solution.** Consider the measure $\frac{dx}{x^{1-\alpha}} \cdot \frac{dy}{y^{1-\beta}}$ on the unit square $[0,1]^2$.

For positive $\alpha$ and $\beta$, $\int_0^1 \int_0^1 \frac{dx}{x^{1-\alpha}} \cdot \frac{dy}{y^{1-\beta}} = \alpha x^\alpha \Big|_0^1 \cdot \beta y^\beta \Big|_0^1 = \alpha\beta$. So, each function which is continuous on the unit square and therefore bounded, can be integrated with this measure. So, for polynomials we may define a scalar product $\langle f, g \rangle = \int_0^1 \int_0^1 f(x,y) \cdot g(x,y) \frac{dx}{x^{1-\alpha}} \cdot \frac{dy}{y^{1-\beta}}$ which is positive-definite, since no nonzero polynomial is zero on the entire square. Let $p_i(x,y) = x^i y^i$, for $i = 1,\dots,n$. This polynomials are linearly independent. Then $\langle p_i, p_j \rangle = \int_0^1 \int_0^1 x^{i+j} \cdot y^{i+j} \frac{dx}{x^{1-\alpha}} \cdot \frac{dy}{y^{1-\beta}} = \frac{1}{i+j+\alpha} \cdot \frac{1}{i+j+\beta}$ which is precisely the entry of the given matrix. So our matrix is a matrix of scalar product, so it is positive definite.

**5.** Let $S$ be the surface area of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Prove that $\frac{4}{3}\pi \le \frac{S}{ab+bc+ca} \le 2\pi$.

**Solution.** For any vector $(x,y,z)$ the length of the vector is bounded by sum of its projections to the 3 axes: $\sqrt{x^2+y^2+z^2} \le |x|+|y|+|z|$. That can be easily seen by taking the square. Equality is achieved when the vector is in a direction of one of the axes.

Therefore, for any planar polygon in space, its area is bounded from above by sum of its projections to the 3 coordinate planes: $xy$, $xz$ and $yz$ (since projecting to a plane requires multiplication to the same cosine, as projecting the vector orthogonal to the polygon to a complimentary axis).

So, by integrating the previous statement, for any surface in space its area is bounded from above by sum of its projections to the 3 coordinate planes, counted with multiplicities (which means that the area of the projection which is covered $k$ times should be taken into account with coefficient $k$). In our case of ellipsoid, projections are ellipses with semiaxes $a$ and $b$, $a$ and $c$, $b$ and $c$, all covered twice, hence

$$s \le 2(\pi ab+\pi ac+\pi bc)$$

The inequality approaches equality when normal vector at almost all points becomes the vector of one of the axes, for instance when $c$ is much smaller than $a$ and $b$.

The second estimate for the surface area comes from considering $\varepsilon$-neighborhood of the ellipsoid. For small $\varepsilon$, the volume of the set of points which are outside the ellipsoid but at distance no more than $\varepsilon$ from it, is $S\varepsilon+o(\varepsilon)$. Denote by $E(a,b,c)=\left\{(x,y,z)\,\middle|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$ the ellipsoid with semi-axes $a,b,c$. Its volume is known to be $\frac{4}{3}\pi abc$.

Now consider $E(a+\varepsilon,b+\varepsilon,c+\varepsilon)$. We want to explain that is within (the closed) $\varepsilon$-neighborhood of $E(a,b,c)$.

Indeed, every point in the $E(a+\varepsilon,b+\varepsilon,c+\varepsilon)$ can be presented as $\begin{pmatrix}(a+\varepsilon)x\\(b+\varepsilon)y\\(c+\varepsilon)z\end{pmatrix}=\begin{pmatrix}ax\\by\\cz\end{pmatrix}+\varepsilon\begin{pmatrix}x\\y\\z\end{pmatrix}$, where $x^2+y^2+z^2$. The first summand is within $E(a,b,c)$, the second is of length at most $\varepsilon$. So,

$$Vol\left(E(a+\varepsilon,b+\varepsilon,c+\varepsilon)\right)-Vol\left(E(a,b,c)\right)\le S\varepsilon+o(\varepsilon)$$

$$\frac{4}{3}\pi\left((a+\varepsilon)(b+\varepsilon)(c+\varepsilon)-abc\right)\le S\varepsilon+o(\varepsilon)$$

$$\frac{4}{3}\pi\left(\varepsilon ab+\varepsilon ac+\varepsilon bc+o(\varepsilon)\right)\le S\varepsilon+o(\varepsilon)$$

$$\frac{4}{3}\pi\left(\varepsilon ab+\varepsilon ac+\varepsilon bc\right)\le S\varepsilon\,.$$

**6.** Let $K(n)$ be the greatest possible number of summands in the representation $n = a_1 + a_2 + a_3 + ... + a_k$, such that $a_1 < a_2 < a_3 < ... < a_k$ are positive integers and $a_1 | a_2 | a_3 | ... | a_k$. Then there exists $C > 0$ such that $K(n) > C\sqrt{\log n}$ for any $n$.

**Solution.** Let $q$ be the minimal natural number, by which $n$ is not divisible.

The we can divide with remainder $n = sq + a$, where $a$ is the remainder.

Then take $n - a$ which is divisible by both $a$ and $q$.

We declare $a = a_1$, replace $n$ by $\frac{n-a}{lcm(a,q)}$, and repeat this process as long as possible.

How long does this process last?

We shall use a version of Prime Number Theorem: $\sum_{\substack{p,k \\ p \text{ prime} \\ \mathrm{p}^k < m}} \log p \sim m$.

It can be reformulated as follows: $\log(\mathrm{lcm}(1,2,3,...,m-1)) \sim m$.

Notice that $n$ is divisible by and hence greater than $\mathrm{lcm}(1,2,...,q-1)$, so $\log(aq) < 2\log q < 2\log\log n$. Hence dividing $n$ by $aq$ (in logarithmic measure, for instance how many digits were erased) is decreasing $\log n$ by at most $2\log\log n$. This bound gets stronger as $n$ gets smaller. So we have at least $\frac{\log n}{2\log\log n} \gg \sqrt{\log n}$ steps.