

Second stage of Israeli students competition, 2017.

Please try to write your solutions in English.

Duration: 4 hours

את השאלון ניתן לקחת איתך בסיום התחרות.

1. A permutation is a bijective function $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$.

A permutation is called even if $\prod_{i < j} (\sigma(j) - \sigma(i)) > 0$ and odd otherwise.

For any permutation σ define its displacement $D(\sigma) = \prod_{i=1}^n |i - \sigma(i)|$.

Which is greater: sum of displacement of all even permutations or sum of displacements of all odd permutations? The answer might depend on n .

2. Prove that $\frac{(n+2)^{n+1}}{(n+1)^n} - \frac{(n+1)^n}{n^{n-1}} < e$, for every positive integer n .

3. Is it possible to find a broken line (not necessary closed) inside the unit square in plane, such that its length is 10000, and each triangle with vertices on the line has

(a) an angle which is less than 15° ?

(b) two angles which are less than 15° ?

4. Let α and β be positive numbers. We construct a symmetric matrix, such that at column i row j the entry is $\frac{1}{i+j+\alpha} \cdot \frac{1}{i+j+\beta}$. Show that this matrix is positive definite.

5. Let S be the surface area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Prove that $\frac{4}{3}\pi \leq \frac{S}{ab+bc+ca} \leq 2\pi$.

6. Let $K(n)$ be the greatest possible number of summands in the representation $n = a_1 + a_2 + a_3 + \dots + a_k$, such that $a_1 < a_2 < a_3 < \dots < a_k$ are positive integers and $a_1 | a_2 | a_3 | \dots | a_k$. Then there exists $C > 0$ such that $K(n) > C\sqrt{\log n}$ for any n .

Good luck!