First stage of Israeli students competition, 2016 – solutions.

1. Compute $\det\begin{pmatrix}1&2&2&2&2\\1&1&2&2&3\\1&1&0&3&3\\1&4&4&3&3\\4&4&4&4&3\end{pmatrix}$.

Answer. 0.

First solution. Subtract the second column from the first and the fourth from the fifth. We get $\det\begin{pmatrix}-1&2&2&2&0\\0&1&2&2&1\\0&1&0&3&0\\-3&4&4&3&0\\0&4&4&4&-1\end{pmatrix}$. Now subtract column 3 from column 4 and from column 2. We get $\det\begin{pmatrix}-1&0&2&0&0\\0&-1&2&0&1\\0&1&0&3&0\\-3&0&4&-1&0\\0&0&4&0&-1\end{pmatrix}$.

Add last row to second row and subtract 3 times first row to fourth row. We get $\det\begin{pmatrix}-1&0&2&0&0\\0&-1&6&0&0\\0&1&0&3&0\\0&0&-2&-1&0\\0&0&4&0&-1\end{pmatrix}=\det\begin{pmatrix}-1&6&0\\1&0&3\\0&-2&-1\end{pmatrix}$. We reduce to a matrix 3×3 since in the first column the only nonzero entry is at the first and in the last column the only nonzero entry is at the last. In the 3×3 matrix add the first row to the second row, and get $\det\begin{pmatrix}-1&6&0\\0&6&3\\0&-2&-1\end{pmatrix}$ and now the middle row is -3 times the last row, so $\det=0$.

Second solution. We shall prove a more general claim, i. e.

$$\det\begin{pmatrix} a & b & b & b & b \\ a & a & b & b & c \\ a & a & 0 & c & c \\ a & d & d & c & c \\ d & d & d & d & c \end{pmatrix} = 0$$

If one would write down all the permutations, he'd get sum of monomials of type $a^m b^i c^n d^k$, where $0 \le m,i,n,k \le 2$, however the sum would contain about 100 monomials, and it can lead to computational mistakes.

Notice, that if a 5×5 matrix is rotated by 90° degrees, its determinant is preserved. Indeed, it is the same as transpose + reversing the order of rows. Transpose preserves the determinant, and reversing the order of rows is a permutation $(15)(24)$ which is even. Therefore the formula should be symmetric w. r. t. $a \to b \to c \to d \to a$ cyclic permutation. For instance, if one computes the coefficient of a^2b^2c, the coefficient of b^2c^2d should be the same.

Let's think about the coefficient of a^2b^2c. From the last row we have to take c so it is the same as coefficient of a^2b^2 in $\det\begin{pmatrix} a & b & b & b \\ a & a & b & b \\ a & a & & \\ a & & & \end{pmatrix}$ (we delete last row and column, and skip all d's), which is zero, as two rightmost columns are the same.

Let's think about the coefficient of a^2b^2d. To avoid c, we take the top entry in the rightmost column, so it is the coefficient of a^2bd in

$$\det\begin{pmatrix} a & a & b & b \\ a & a & & \\ a & d & d & \\ d & d & d & d \end{pmatrix} = \det\begin{pmatrix} a & a & 0 & b \\ a & a & & \\ a & d & d & \\ d & d & 0 & d \end{pmatrix}$$

Here the fourth column was subtracted from the third. So we take d form the third column, one more b from the first row, and two a's have to come from the second row, but they can't.

Let's think about the coefficient of $a^2 \cdot b \cdot c^2$. To avoid d, we the last entry in the last row, so it is the same as the coefficient of a^2bc in

$$\det\begin{pmatrix} a & b & b & b \\ a & a & b & b \\ a & a & & c \\ a & & & c \end{pmatrix} = \det\begin{pmatrix} a & b & b & b \\ a & a & b & b \\ 0 & a & & 0 \\ a & & & c \end{pmatrix}$$

Here we've subtracted the 4th row from the third. We should take the remaining c in the corner and the only nonzero a in the third line and ab from the remaining 2×2 matrix $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ but here the terms cancel each other.

We have analyzed monomials a^2b^2c, a^2b^2d and a^2bc^2 which by rotational symmetry give all monomials of with powers $2,2,1$ and all coefficients are zeroes. The remaining monomials have powers $2,1,1,1$ and are all symmetric to each other by the same rotational symmetry. So the determinant is given by the formula

$$K \cdot \left(a^2bcd + ab^2cd + abc^2d + abcd^2\right)$$

where K is a constant which is yet to be determined. One could play one more game of Sudoku, but it is easier to substitute $a = b = c = d$. The determinant in this case is $4K$, but on the other hand, the matrix has 4 equal rows so it is surely degenerate. Therefore $4K = 0$, and determinant is always zero.

2. Which curve has greater length: the ellipse $\left\{(x, y) \,\middle|\, \frac{x^2}{2} + y^2 = 1\right\}$ or one wave of the sine $\{(x, \sin x) \mid 0 \le x \le 2\pi\}$?

Answer. They are equal.

First solution. One can try to compute both. The ellipse is easier described parametrically $\left(\sqrt{2}\cos t, \sin t\right)$ and the length of infinitesimal piece can be computed by Pythagoras theorem:

$$\sqrt{dx^2+dy^2}=\sqrt{2\sin^2 t\cdot dt^2+\cos^2 t\cdot dt^2}=\sqrt{2\sin^2 t+\cos^2 t}\cdot dt=$$
$$=\sqrt{\sin^2 t+1}\cdot dt$$

The length is then $\int\limits_0^{2\pi}\sqrt{\sin^2 t+1}\cdot dt$. That is not the easiest integral to compute, so let us meanwhile write the other one. By the same application of Pythagoras theorem, we find the infinitesimal piece

$$\sqrt{dx^2+dy^2}=\sqrt{dx^2+\left(\cos x\cdot dx\right)^2}=\sqrt{1+\cos^2 x}\cdot dx.$$

And the length is $\int\limits_0^{2\pi}\sqrt{1+\cos^2 x}\cdot dx$ which is definitely the same as before (just take $t=x+\frac{\pi}{2}$).

Remark. Those who like calculus, can write the answer as an elliptic integral of the second kind.

Second solution. Take a sheet of paper on which the graph of the sine appears, and roll it into a cylinder of radius 1, by gluing the line $\{x=0\}$ to $\{x=2\pi\}$ at the same y. The graph will become a planar ellipse, the plane cutting the cylinder at the angle of 45°. The half-axes of this ellipse are $\sqrt{2}$ and 1. The length is preserved in the process, so the lengths are equal.

3. Compute $\lim\limits_{n\to\infty}\left(\frac{3}{2^2}-\frac{1}{3^2}+\frac{3}{4^2}-\frac{1}{5^2}+....-\frac{1}{(2n-1)^2}+\frac{3}{(2n)^2}\right)^n$.

Answer. $\frac{1}{\sqrt{e}}$.

Solution.

$$a_n=\left(\frac{3}{2^2}-\frac{1}{3^2}+\frac{3}{4^2}-\frac{1}{5^2}+....-\frac{1}{(2n-1)^2}+\frac{3}{(2n)^2}\right)^n=$$

$$=\left(1-1+\frac{3}{2^2}-\frac{1}{3^2}+\frac{3}{4^2}-\frac{1}{5^2}+....-\frac{1}{(2n-1)^2}+\frac{3}{(2n)^2}\right)^n=$$

$$=\left(1-\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{(2n)^2}\right)+4\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+...+\frac{1}{(2n)^2}\right)\right)^n=$$

$$=\left(1-\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{(2n)^2}\right)+\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}\right)\right)^n=$$

$$=\left(1-\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+...+\frac{1}{(2n)^2}\right)\right)^n$$

Notice that

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+...+\frac{1}{(2n)^2}\le$$

$$\le\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+...+\frac{1}{(2n-1)(2n)}=\frac{1}{n}-\frac{1}{2n}=\frac{1}{2n}$$

But

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+...+\frac{1}{(2n)^2}\ge$$

$$\ge\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+...+\frac{1}{(2n)(2n+1)}=\frac{1}{n+1}-\frac{1}{2n+1}>$$

$$=\frac{n}{(n+1)(2n+1)}=\frac{1}{2n+3+\frac{1}{n}}>\frac{1}{2n+4}$$

So $\left(1-\frac{1}{2n+4}\right)^n<a_n<\left(1-\frac{1}{2n}\right)^n$.

It is classical that $\lim\left(1-\frac{1}{2n+4}\right)^{-2n-4}=e=\lim\left(1-\frac{1}{2n}\right)^{-2n}$.

Therefore $\lim\left(1-\frac{1}{2n}\right)^n = \lim\left(\left(1-\frac{1}{2n}\right)^{-2n}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$ and

$$\lim\left(1-\frac{1}{2n+4}\right)^n = \lim\left(\left(1-\frac{1}{2n+4}\right)^{-2n-4}\right)^{\frac{n}{-2n-4}} =$$

$$= \left(\lim\left(1-\frac{1}{2n+4}\right)^{-2n-4}\right)^{\lim\left(\frac{n}{-2n-4}\right)} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

Hence $\frac{1}{\sqrt{e}} = \lim_{n\to\infty}\left(1-\frac{1}{2n+4}\right)^n \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n = \frac{1}{\sqrt{e}}$

So the limit is $\frac{1}{\sqrt{e}}$.

4. In a village there are 100 people. Some of them are friends. The friendship is mutual (if Avi is a friend of Beni, Beni is a friend of Avi) but not transitive (if Avi is a friend of Beni, and Beni is a friend of Gadi, Gadi doesn't have to be a friend of Avi). Once a while, one of the people may decide to start a new life: he stops being friends with all his friends, and becomes a friend of all others. This can happen several times.

Is it always possible the villagers behave in such a way, so that no matter what was the original situation, in the end

(a) At least 50.5% of all pairs of people will be friends?

(b) At least 51% of all pairs of people will be friends?

Answer. (a) Yes. (b) No.

Solution. (a) If someone has more enemies than friends, he can change his situation and increase the number of friendships. The number of friendships cannot be increased too many times, so at some moment each will have at least 50 friends (out of 99), so the total number of friendly pairs is at least $\frac{50}{99}$ out of all airs. It remains to show that $\frac{50}{99} > \frac{50.5}{100} = \frac{101}{200}$, i. e. $50 \cdot 200 > 99 \cdot 101$, or $10000 > 9999$.

(b) We shall describe a situation, in which it is impossible. Assume there are two kinds of people: 50 red and 50 blue, and the friendly couples are people of different kind. It is possible to do some changes, but after several steps all situations which can arise can be described in a similar way, simply some people change their color. So, in the end there are $50+k$ red people and $50-k$ blue (here k is an integer, but not necessary nonnegative), and $(50+k)(50-k)=50^2-k^2$ friendships, no more than original 50^2. The total number of pairs is $\frac{100\cdot 99}{2}=50\cdot 99$. The maximal share of friendships in all pairs is $\frac{50}{99}$, which is not as much as needed, as

$$\frac{50}{99}<\frac{51}{100}$$

(in general the fraction $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$, the easiest way to understand it is to merge orange juice of different concentration; and hence $\frac{50+1}{99+1}$ is between $\frac{50}{99}$ and $\frac{1}{1}$, which is greater).

5. Let $a_n=2^n+3^n+5^n$. Show that there exists n such that a_n has 5777 different prime divisors.

Solution. We shall prove by induction on k that there is n such that a_n has k different prime divisors. Firstly, it is easy to see that for each prime number p the sequence is periodic modulo p (at least for $n>1$). Therefore, if there are k distinct prime numbers $p_1, p_2, ..., p_k$ such that a_n is divisible by $p_1 p_2 ... p_k$, then also $a_{n+c}, a_{n+2c}, a_{n+3c}, ...$ are also divisible by $p_1 p_2 ... p_k$, where c is any common multiple of all periods of the sequence a_n modulo p_i. We would like to prove that a_{n+mc} at least for some m is divisible by one more prime number.

If not, each a_{n+mc} is of form $p_1^{\ell_1} p_2^{\ell_2} \cdot ... \cdot p_k^{\ell_k}$. Among the k factors $p_1^{\ell_1}$, $p_2^{\ell_2}$, . . . , $p_k^{\ell_k}$ one is the greatest; let us say $p_j^{\ell_j}$; in this case we shall say that a_{n+mc} is of color j. That way, each a_{n+mc} has a color which has one of k possible values.

Among $2k+1$ consequent values of m, in $a_{n+Mc}, a_{n+(M+1)c}, \ldots, a_{n+(M+2k)c}$, there will be 3 numbers of the same color by pigeon-hole.

Those 3 numbers m_1, m_2, m_3 have a large common divisor, at least as large as $\sqrt[k]{a_{n+Mc}} > \sqrt[k]{5^{Mc}}$. So given n,c,k we can take as large M as we wish, and so find $n_i = n + m_i c$ such that $a_{n_1}, a_{n_2}, a_{n_3}$ have a huge common divisor, greater than $5^{\frac{c}{k}M}$. Now we shall show that this greatest common divisor cannot be too great, and so obtain a contradiction.

Assume $m_1 < m_2 < m_3$, then $m_2 = m_1 + s$, $m_3 = m_1 + t$, and $t > s$, and also denote $n + m_1 c = N$. We consider the numbers a_N, a_{N+sc}, a_{N+tc}, or

$$2^N + 3^N + 5^N \ , \ 2^{N+sc} + 3^{N+sc} + 5^{N+sc} \ , \ 2^{N+tc} + 3^{N+tc} + 5^{N+tc}$$

Their common divisor is also a common divisor of (subtract from the last two numbers powers of 5 times the first number)

$$\left(2^{sc} - 5^{sc}\right)2^N + \left(3^{sc} - 5^{sc}\right)3^N \ , \ \left(2^{tc} - 5^{tc}\right)2^N + \left(3^{tc} - 5^{tc}\right)3^N$$

Their common divisor is also a divisor of (take $\left(3^{sc} - 5^{sc}\right)$ the second number minus $\left(3^{tc} - 5^{tc}\right)$ the first number) to

$$\left(\left(3^{sc} - 5^{sc}\right)\left(2^{tc} - 5^{tc}\right) - \left(2^{sc} - 5^{sc}\right)\left(3^{tc} - 5^{tc}\right)\right)2^N = 2^N \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ 2^{tc} & 3^{tc} & 5^{tc} \\ 2^{sc} & 3^{sc} & 5^{sc} \end{pmatrix}$$

We could run this computation differently (exclude powers of 2 first and not powers of 5) and we would get that the common divisor is a divisor of

$$5^N \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ 2^{tc} & 3^{tc} & 5^{tc} \\ 2^{sc} & 3^{sc} & 5^{sc} \end{pmatrix}$$

So, anyway, the common divisor is a divisor of $D = \det\begin{pmatrix} 1 & 1 & 1 \\ 2^{tc} & 3^{tc} & 5^{tc} \\ 2^{sc} & 3^{sc} & 5^{sc} \end{pmatrix}$.

This determinant is nonzero. The simplest explanation I know is that the planar curve $\left\{\left(x, x^{\alpha}\right) \middle| x > 0\right\}$ is convex, where $\alpha = \frac{t}{s}$, so 3 different points on this curve cannot be on one line (but I believe people who know well Schur's polynomials can think of easier explanation). Anyway, $|D| < 6\cdot 5^{(t+s)c}$ (six permutations, each is less than $5^{(t+s)c}$) so any common divisor of a_N, a_{N+sc}, a_{N+tc} is less than $6\cdot 5^{(t+s)c}$, but t and s are at most $2k$ so the common divisor is at most $6\cdot 5^{4kc}$, but is greater than $5^{\frac{c}{k}M}$, where M is any prim number. That is not possible, so eventually we have an extra prime divisor.

6. Let $p(x)$ and $q(x)$ be polynomials with nonnegative coefficients, with leading coefficient 1. Let $p(x)\cdot q(x) = 1 + x + x^2 + x^3 + ... + x^k$. Show that all coefficients of $p(x)$ and $q(x)$ are zeroes or ones.

Solution. Both $p(x)$ and $q(x)$ are with real coefficients, so with each root each polynomial has its complex conjugate. But all roots we have are of absolute value 1, so we could reformulate the same statement differently: if z is a root of $p(x)$ or $q(x)$, then $\frac{1}{z}$ is a root of the same polynomial (and of the same multiplicity, since all roots are of multiplicity. So if $p(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_2x^2 + a_1x + a_0$ then the sequence a_j is symmetric, $a_j = a_{m-j}$, and also if $q(x) = b_n x^n + ... + b_1 x + b_0$ then $b_j = b_{n-j}$. Of course, here $m + n = k$.

We want to prove that all coefficients of $p(x)$ and $q(x)$ are zeroes or ones. Assume the opposite. Of course, coefficients cannot be greater than 1: if $a_j > 1$ then the coefficient of x^{n+j} of $p\cdot q$ is a_j plus nonnegative, but it should be one. Notice that $a_0 = 1 = b_0$ since they are at most 1, both nonnegative, and their product is 1.

Let t be the smallest power of x, which has a coefficient which is not 0 or 1, say $0 < a_\ell < 1$ and $h = m - t$ is the smallest possible; WLOG it happens with $p(x)$. So we can assume that all coefficients of lower power in both p and q are ones or zeroes, that includes $a_0, a_1, ..., a_{\ell-1}$ and $b_0, b_1, ..., b_{\ell-1}$.

Consider the coefficient of ℓ of $p \cdot q$ (which is 1 as all coefficients the product) it consists of $1 \cdot a_\ell + b_\ell \cdot 1 +$ products of ones and zeroes.

So $a_\ell + b_\ell$ is integer, hence $b_\ell = 1 - a_\ell$, so $0 < b_\ell < 1$. By symmetry we get $b_{n-\ell} = b_\ell$, so $0 < b_{n-\ell} < 1$.

Now consider the coefficient of x^n in $p \cdot q$. It has $b_{n-\ell} a_\ell$ which is strictly positive and $b_n a_0 = 1$ as summands, and more nonnegative summands. That is a contradiction.