First stage of Israeli students competition, 2017.

Please try to write your solutions in English.

Duration: 4 hours

את השאלון ניתן לקחת איתך בסיום התחרות.

1. Compute $\det\begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 0 & 3 & 3 \\ 1 & 4 & 4 & 3 & 3 \\ 4 & 4 & 4 & 4 & 3 \end{pmatrix}$.

2. Which curve has greater length: the ellipse $\left\{(x,y)\,\middle|\,\frac{x^2}{2}+y^2=1\right\}$ or one wave of the sine $\left\{(x,\sin x)\,\middle|\,0\le x\le 2\pi\right\}$?

3. Compute $\lim\limits_{n\to\infty}\left(\frac{3}{2^2}-\frac{1}{3^2}+\frac{3}{4^2}-\frac{1}{5^2}+\frac{3}{6^2}-\frac{1}{7^2}+\ldots-\frac{1}{(2n-1)^2}+\frac{3}{(2n)^2}\right)^n$.

4. In a village there are 100 people. Some of them are friends. The friendship is mutual (if Avi is a friend of Beni, Beni is a friend of Avi) but not transitive (if Avi is a friend of Beni, and Beni is a friend of Gadi, Gadi might be not a friend of Avi). Once a while, one of the people may decide to start a new life: he stops being friends with all his current friends, and becomes a friend of all others. Such events may happen several times.

Can the villagers behave in such a way, so that no matter what was the original situation, in the end:

(a) At least 50.5% of all pairs of people will be friends?

(b) At least 51% of all pairs of people will be friends?

5. Let $a_n=2^n+3^n+5^n$. Show that there exists n such that a_n has at least 5777 distinct prime divisors.

6. Let $p(x)$ and $q(x)$ be polynomials with nonnegative real coefficients, with leading coefficient 1. Let $p(x)\cdot q(x)=1+x+x^2+x^3+\ldots+x^k$. Show that all coefficients of $p(x)$ and $q(x)$ are either zeroes or ones.

Good luck!