Second stage of Israeli students competition, 2016.
Please try to write your solutions in English.
Duration: 4 hours

1. Let $f:[0,1]\to[e,+\infty)$ be a monotonically increasing function. Prove that there exist $x,y\in[0,1]$ such that $f(y)\le 2f(x)$ and $y-x\ge\dfrac{1}{10\left(\ln f(x)\right)^2}$.

Solution. Assume the contrary. Then for $y=x+\dfrac{1}{10\cdot\left(\ln f(x)\right)^2}$ (if $y\le 1$) we get $f(y)\ge 2f(x)$. Or, if we define $g:[0,1]\to[1,+\infty)$, $g(x):=\ln\left(f(x)\right)$, then for $y=x+\dfrac{1}{10\cdot\left(g(x)\right)^2}$ we get $g(y)\ge g(x)+\ln 2$.

Define a sequence: $x_0=0$, $x_{n+1}=x_n+\dfrac{1}{10\cdot\left(g(x_n)\right)^2}$. Then, as long as the sequence is defined, $g(x_n)\ge 1+n\cdot\ln 2$, and hence

$$x_n=\sum_{k=0}^{n-1}\frac{1}{10\cdot\left(g(x_k)\right)^2}\le\sum_{k=0}^{n-1}\frac{1}{10\cdot(1+k\cdot\ln 2)^2}\le\sum_{k=0}^{n-1}\frac{1}{(3+3k\cdot\ln 2)^2}\le\sum_{k=0}^{n-1}\frac{1}{(3+k\cdot\ln 8)^2}\le$$
$$\le\sum_{k=0}^{n-1}\frac{1}{(3+k)^2}<\sum_{k=0}^{n-1}\frac{1}{(2+k)(3+k)}=\sum_{k=0}^{n-1}\left(\frac{1}{(2+k)}-\frac{1}{(3+k)}\right)=\frac{1}{2}-\frac{1}{2+n}<\frac{1}{2}$$

as long as x_n is defined. But that means x_n is always well defined, since if $x_n<1$ then $g(x_n)$ is also defined, and x_{n+1} is well defined, and we have even proved that $x_n<\frac{1}{2}$. To summarize: we proved existence of x_n such that $f(x_n)\ge 2^n f(0)\ge e\cdot 2^n$ and therefore $f(x_n)\to+\infty$, but $f(x_n)\le f\left(\frac{1}{2}\right)$ which is a contradiction.

2. Prove that for every irrational number $\alpha\in(0,1)$, there exists a non-decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of positive integers, such that $\alpha=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{a_n a_{n+1}}$.

Proof. Recall that continued fraction produces for each irrational number α an infinite sequence of rational approximations $\frac{p_n}{q_n}$, which is rapidly converging to α, such that the first element is $\frac{p_0}{q_0}=\frac{1}{q_0}$, and $\frac{p_k}{q_k}-\frac{p_{k-1}}{q_{k-1}}=\frac{(-1)^k}{q_k q_{k-1}}$, the sequence $\{q_i\}$ is a strictly increasing sequence of positive integers and therefore

$$\alpha=\lim_{n\to\infty}\frac{p_n}{q_n}=\lim_{n\to\infty}\left(\frac{p_0}{q_0}+\sum_{k=0}^{n-1}\left(\frac{p_{k+1}}{q_{k+1}}-\frac{p_k}{q_k}\right)\right)=\lim_{n\to\infty}\left(\frac{1}{q_0}+\sum_{k=1}^{n}\frac{(-1)^k}{q_{k-1}q_k}\right)=$$
$$=\frac{1}{1\cdot q_0}-\frac{1}{q_0\cdot q_1}+\frac{1}{q_1\cdot q_2}-\frac{1}{q_2\cdot q_3}+\frac{1}{q_3\cdot q_4}-\frac{1}{q_4\cdot q_5}+\ldots$$

Q.E.D.

3. Numbers $a_1,...,a_{16}$ are written in this order around a circle. At each move, all numbers are simultaneously replaced: a_i is replaced by $|a_i-a_{i-1}|$ (we take $a_0=a_{16}$ since the order is circular). Is it true that after finite number of moves, all numbers around the circle will become zeroes, assuming that
(a) the numbers are rational?
(b) the numbers are real?

Answers. (a) yes, (b) no.

Solution. (a) We can multiply all fractions numbers by their common denominator, and play similar (but equivalent) game with integer numbers. Starting with the second move, all numbers are non-negative; so from now on we shall assume all numbers around the circle are nonnegative.

Let $M=\max a_i$. Then also $|a_i-a_{i+1}|\le M$ (for non-negative numbers). So $a_i<M$ throughout the game. If there is a common divisor d for all a_i, we can divide all numbers by d and consider an equivalent game with smaller M. We shall prove that **after 16 moves all the numbers in the circle become even**. If that is true,

then after 16 moves we can replace M by $\frac{M}{2}$. Applying this idea several times, we see that after $16\left(1+\left[\log_2 M\right]\right)$ all numbers are zeroes.

So, to prove that all numbers become even after 16 moves, we consider the process mod 2. Then the rules are more simple: a_i is replaced by $a_i + a_{i+1}$. Iterating this process, we get that after k moves the number at place i is $\sum_j \binom{k}{j} a_{i-j}$.

Recall that $\binom{16}{j} = 0 \pmod 2$ for $0 < j < 16$ (if you didn't know it, prove it by using 4 times the identity $(1+x)^2 = 1 + x^2 \pmod 2$). Therefore after precisely 16 moves, we get that at place i we get a number $\binom{16}{0} a_i + \binom{16}{16} a_i = a_i + a_i = 0 \pmod{16}$, which completes the proof.

Remark. Here it is important that 16 is a power of 2. Otherwise it is wrong.

(b) Let us write around the circle the geometric sequence: $1, x, x^2, x^3, ..., x^{15}$ in this order (we shall choose $x > 1$ later). Then after one move we get

$$x-1, x^2 - x, ..., x^{15} - x^{14}, x^{15} - 1$$

or in other words

$$z, zx, zx^2, ..., zx^{14}, x^{15} - 1$$

where $z = x - 1$ is a positive number. If also $x^{15} - 1 = z\left(x^{16} - x\right)$, then all the numbers during this move were simply multiplied by z. So, from now on with each move all the numbers will be multiplied by z, and will remain nonzero forever. So, it remains to choose $x > 1$, such that $x^{15} - 1 = (x-1)\left(x^{16} - 1\right)$. Which means $x^{14} + x^{13} + x^{12} + ... + 1 = x^{16} - 1$, or that x is a root of the polynomial

$$x^{14} + x^{13} + x^{12} + ... + 2 - x^{16}.$$

The polynomial is positive for $x = 1$ and negative for $x = 2$ so there has to be a root somewhere in between.

4. Prove that for each $a,b,c,d \in \mathbb{R}$, the following inequality holds:

$$\sqrt{a^2+b^2+\tfrac{2}{3}ab}\cdot\sqrt{c^2+d^2+\tfrac{2}{3}cd}+\sqrt{b^2+c^2+\tfrac{2}{3}bc}\cdot\sqrt{a^2+d^2+\tfrac{2}{3}ad} \geq \sqrt{a^2+c^2+\tfrac{2}{3}ac}\cdot\sqrt{b^2+d^2+\tfrac{2}{3}bd}.$$

Solution. Consider a regular tetrahedron in $\mathbb{R}^3$, inscribed in a sphere of unit radius, with the center at the origin. The vertices of the tetrahedron are represented by vectors v_1, v_2, v_3, v_4. Then $v_1+v_2+v_3+v_4=0$, and hence

$$0=\left\langle \sum_i v_i, \sum_j v_j \right\rangle = \sum_{i=j}\left\langle v_i, v_j \right\rangle + \sum_{i\neq j}\left\langle v_i, v_j \right\rangle = 4+12\left\langle v_1, v_2 \right\rangle,$$

so cosine of the angle between each two different vectors is $-\frac{1}{3}$.

Consider 4 points in $\mathbb{R}^3$: $A=av_1$, $B=bv_2$, $C=cv_3$, $D=dv_4$. Ptolemy inequality states that for each 4 points, $AB\cdot CD+BC\cdot AD\geq AC\cdot BD$. In our case all the lengths might be expressed with cosine theorem, for example $AB^2=a^2+b^2+\frac{2}{3}ab$. Substitute all such expressions in Ptolemy inequality, Q. E. D.

5. Show that the polynomial $x^n+x^{n-1}+x^{n-2}+\ldots+x^2+x-1$ is irreducible over $\mathbb{Q}$.

Proof. After reversing the coefficients we reduce to showing that $h(x)=x^n\cdot f\left(\frac{1}{x}\right)$ is irreducible. By the lemma of Gauss it is sufficient to show irreducibility over $\mathbb{Z}$. Any factorization of h over $\mathbb{Z}$ determines a partition of the roots of h into two sets such that the product of each set is a nonzero integer (the constant term of the corresponding factor), and hence has norm ≥ 1. Thus, each of these sets must contain an element of norm ≥ 1. Therefore, if we show that h has at most one root of norm ≥ 1 then no factorization can exist and we are done. For that it is sufficient to show that f has at most one root of norm ≤ 1. Observe that $f(x)(x-1)=x^{n+1}-2x+1$ so every root of f is a fixed point of the complex mapping $T(z)=\dfrac{z^{n+1}+1}{2}$.

Suppose that f has two distinct roots z_1, z_2 of norm ≤ 1. If for some $i\in\{1,2\}$ we have $|z_i|=1$ then

$$1=\left|z_i\right|=\left|\frac{z_i^{n+1}+1}{2}\right|\le\frac{\left|z_i^{n+1}\right|+1}{2}\le\frac{1+1}{2}=1$$

so we have equality in the triangle inequality which means that $z_i^{n+1}\in\mathbb{R}$ which means that $z_i=T\left(z_i\right)\in\mathbb{R}$ but $f\left(\pm1\right)\neq0$. So z_1,z_2 are two distinct fixed points of the map T which maps the open unit disk to itself. By the equality case of the Schwarz-Pick theorem (or as a consequence of Schwarz lemma) T is a Moebius transformation which is a contradiction (unless $n=1$, but in this case irreducibility is obvious).

The proof is complete, but for those who don't remember it we shall remind the proof of Schwarz-Pick. We want to show that if there is a complex analytic mapping from the open unit disc to itself with two fixed internal points, then it is identity. By conjugation with some Moebius transformation, we may assume that one of the fixed points is 0, and another will be denoted by z_0.

So, we have a complex analytic function in the open unit disc $\Delta=\left\{z\,\middle|\,\left|z\right|<1\right\}$, satisfying $f\left(0\right)=0$, $\left|f\left(z_0\right)\right|=\left|z_0\right|$ for some $z_0\in\Delta$, and $f\left(\Delta\right)\subset\Delta$. Then is possible to define a complex analytic function $g\left(x\right)=\begin{cases}\frac{f\left(z\right)}{z} & z\neq0\\ f'\left(0\right) & z=0\end{cases}$.

It is easy to see that on a circle $\left|z\right|=r$ for each $r<1$ we get $\left|g\right|\le\frac{1}{r}$. Hence by maximum principle in the disc $\left|z\right|\le r$, we get $\left|g\right|\le\frac{1}{r}$. When r tends to 1 from below, we conclude that $\left|g\left(z\right)\right|\le1$ for all $z\in\Delta$. But $\left|g\left(z_0\right)\right|=1$. Hence by maximum principle (non-constant holomorphic function can not have internal maximum) the function g is constant. From this the lemma follows.

6. There are N boys and N girls at the school. Each girl is acquainted with precisely K boys and each boy is acquainted with precisely K girls (the number K is the same for all boys and girls). For any two girls there are precisely C boys

they are both acquainted with. Prove that for any two boys there are precisely C girls they are both acquainted with.

First solution. We shall number the boys from 1 to N, and number the girls from 1 to N. We shall construct and $N \times N$ matrix A as follows: the number at row i column j is 1 if the i'th girl i is acquainted with the j'th boy and zero otherwise.

As usual, let $e_1, e_2, ..., e_n$ be the vectors of the standard basis. Then vectors Ae_i represent (by zeroes and ones) the friends of the i'th boy, and the vectors $A^T e_j$ represent the friends of the j'th girl. All the sentences in the formulation of the problem can be translated to the language of Linear Algebra (which is shorter):

It is given that $\left\langle A^T e_i, A^T e_j \right\rangle$ *is* K *when* $i = j$, *and* C *when* $i \neq j$. *It is also given that* $\left\langle Ae_i, Ae_j \right\rangle$ *is* K *when* $i = j$, *and we must prove it is* C *when* $i \neq j$.

Or even shorter: *the matrix* AA^T *is given: the entries on the main diagonal are all equal* K, *the other entries are all equal* C. *Show that* $A^T A$ *is the same.*

Notice that the vector $u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector of all matrices we've mentioned:

indeed, $Au = Ku$, and $A^T u = Ku$, so it is also eigenvector of $A^T A$ and of AA^T.

Remark. From here it is easy to see the relation between K and C: on one hand, $AA^T u = K \cdot Au = K^2 u$, on the other hand, by direct computation $AA^T u = \left(K + (n-1)C\right)u$, so $K^2 - K = (n-1)C$. This formula might be also obtained by elementary counting in two ways of triples "boy and two girls he knows" (see the third solution). Anyway, we won't use it in this solution.

Let's discuss eigenvalues and eigenvectors of matrix AA^T.

Let us start with matrix U, all entries of which are 1. This matrix is a matrix of rank 1, so 0 is eigenvalue of U of multiplicity $n-1$. Another eigenvalue is n, of multiplicity 1, and the corresponding eigenvector is u. Since matrix is symmetric

the eigenvectors of 0 (a.k.a. the kernel) is $n-1$-dimensional space, consisting of all vectors orthogonal to u.

The matrix $AA^T = C \cdot U + (K-C) \cdot I$ has the same set of eigenvectors, but eigenvalues are different: for u the eigenvalue is K^2, and for its orthogonal complement the eigenvalue is $K-C$.

Recall that for any real square matrix, AA^T is similar to A^TA. Indeed, by SVD decomposition, there are two orthogonal matrices L, R and a diagonal matrix D such that $A = LDR$. Then $AA^T = LDRR^TDL^T = LD^2L^{-1}$, so it is similar to D^2 and also to $A^TA = R^TD^TL^TLDR = R^{-1}D^2R$.

Let us summarize what we know about A^TA by now. It is similar to the A^TA which is known precisely. It is also a symmetric matrix, so it has eigenspaces of maximal allowed dimensions for all eigenvalues, and different eigenspaces are orthogonal to each other. There are two distinct eigenvalues, same as for AA^T: one of multiplicity 1, another of multiplicity $n-1$. The eigenspace for the first eigenvalue is spanned by u, so the eigenspace of the other eigenvalue is its orthogonal complement. So eigenvalues and eigenspaces of A^TA are the same as for AA^T, and hence the matrices are the same.

Second solution. As before, we reformulate the problem with linear algebra. We shall use the same notations as in the previous solution. $AU = KU$, $A^TU = KU$, where U is matrix of ones.

If $K = C$, every two girls share all their acquaintances, so each boy knows either none of them or all of them. In this case, the problem is easy. We shall assume that from now on that $C < K$.

$$\frac{A^T}{K-C} \cdot \left(A - \frac{C}{K} U \right) = \frac{1}{K-C} \cdot \left(A^TA - \tfrac{C}{K} A^TU \right) =$$
$$= \frac{1}{K-C} \cdot \left((K-C)I + \cancel{CU} - \tfrac{C}{\cancel{K}} \cancel{KU} \right) = I$$

So A^T is inverse of $A-\frac{C}{K}U$, but $A^TU=KU=\left(AU\right)^T=U^TA^T$, so A^T commutes with U, so A^T with A. Therefore $A^TA=AA^T$.

Third solution. Let us compute the number *M* of triples

two boys and a girl who knows both.

On one hand, each girl participates in $\binom{K}{2}$ such triples, since among her friends there are $\binom{K}{2}$ pairs of boys, so $M=N\cdot\binom{K}{2}$. On the other hand, if the average number of girls that boys i and j both know in common is $C_{i,j}$, then .

Therefore $$N\cdot\binom{K}{2}=\sum_{i<j}C_{i,j}\,.$$

Similarly, if we compute in two ways number of triples

two girls and a boy who knows both,

we get that $$N\cdot\binom{K}{2}=\binom{N}{2}\cdot C$$

Now let's compute the number of quadruples:

two boys and two girls who know both.

On one hand, any two girls participate in precisely $\binom{C}{2}$ such groups (since we have to choose 2 boys among their common friends. On the other hand, boys i and j participate in precisely $\binom{C_{i,j}}{2}$ such groups. So we get another identity

$$\binom{N}{2}\binom{C}{2}=\sum_{i<j}\binom{C_{i,j}}{2}$$

We have found two identities on the numbers $C_{i,j}$. Both identities are satisfied if $C_{i,j}=C$ for all i, j . But is there any other way to satisfy both identities?

Recall that $\binom{x}{2}$ is a strictly convex quadratic function. Therefore, by Jensen's inequality, given $\sum_{i<j} C_{i,j}$, the value of $\sum_{i<j}\binom{C_{i,j}}{2}$ will be minimal if and only if all $C_{i,j}$ are equal to each other. Therefore, $C_{i,j}=C \ \ \forall i, j$ is a unique solution for both equalities. So for any two boys there are precisely C girls who know both.

7. The set $\mathbb{L}=\mathbb{R}^2\times\left[0,\frac{1}{100}\right]$ (which is a subset of $\mathbb{R}^3$) is colored in 4 colors. Prove that there exist two points in $\mathbb{L}$ which are of the same color, and the distance between them is 1.

Solution. Consider a circle of radius $R>\frac{1}{2}$ in $\mathbb{R}^2$. Construct a sequence of points $A_1, A_2, \ldots$ on the circle such that A_i does not coincide with A_{i+2}, and the length of A_iA_{i+1} is 1. If for some n, the point A_{2n} coincides with A_1, then R will be called ***forbidden radius***. It is easy to see that when n is large enough, then small changing of R causes big changing of the location of A_{2n}. Therefore, the set of forbidden radii is a dense subset of $\left(\frac{1}{2},\infty\right)$.

It is easy to see, that whenever we get a circle of forbidden radius and its point are colored in less than 3 colors, then there have to be two points of the same color at distance 1.

Now take the point $Q=\left(0,0,\frac{1}{1000}\right)$. Let $\mathbb{W}_Q$ be the set of all points $W=(x,y,z)$ satisfying the following three conditions:

- $x^2+y^2<\dfrac{1}{10^9}$
- $\dfrac{1}{200}<z<\dfrac{1}{100}$
- $QW=2\sqrt{1-R^2}$, where R is a forbidden radius.

Consider the circle $\mathbf{C} = \{X \mid XQ = 1 = XW\}$. Since the direction of the interval QW is almost vertical, the plane of the circle is almost horizontal, so $\mathbf{C} \subset \mathbb{L}$. It is easy to see by Pythagoras theorem that $\mathbf{C}$ is of radius R.

Assume there are no points at unit distance of the same color. So the circle $\mathbf{C}$ has points of at least 3 colors. But the colors of points Q and W are different from all colors of $\mathbf{C}$. So Q and W are of the same color (since there are just 4 colors in total), for any $W \in \mathbb{W}_Q$.

Now take a huge natural number n, and take vectors $u = \begin{pmatrix} \frac{1}{2n} \\ 0 \\ z \end{pmatrix}$, $u = \begin{pmatrix} \frac{1}{2n} \\ 0 \\ -z \end{pmatrix}$, such that $\left(\frac{1}{2n}\right)^2 + z^2 = R^2$, $z > 0$. Take the sequence of points $Q_0 = Q$, and then inductively $W_i = Q_i + u$, $Q_{i+1} = W_i + v$. All this points have to be of the same color. So Q_0 and Q_n are of the same color, but the distance between them is 1.