Second stage of Israeli students competition, 2016.
Please try to write your solutions in English.
Duration: 4 hours

1. Let $f:[0,1]\to[e,+\infty)$ be a monotonically increasing function. Prove that there exist $x,y\in[0,1]$ such that $f(y)\le 2f(x)$ and $y-x\ge\dfrac{1}{10\left(\ln f(x)\right)^2}$.

2. Prove that for every irrational number $\alpha\in(0,1)$, there exists a non-decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of positive integers, such that $\alpha=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{a_n a_{n+1}}$.

3. Numbers $a_1,...,a_{16}$ are written in this order around a circle. At each move, all numbers are simultaneously replaced: a_i is replaced by $|a_i-a_{i-1}|$ (we take $a_0=a_{16}$ since the order is circular). Is it true that after finite number of moves, all numbers around the circle will become zeroes, assuming that
(a) the numbers are rational?
(b) the numbers are real?

4. Prove that for each $a,b,c,d\in\mathbb{R}$, the following inequality holds:

$$\sqrt{a^2+b^2+\tfrac{2}{3}ab}\cdot\sqrt{c^2+d^2+\tfrac{2}{3}cd}+\sqrt{b^2+c^2+\tfrac{2}{3}bc}\cdot\sqrt{a^2+d^2+\tfrac{2}{3}ad}\ge\sqrt{a^2+c^2+\tfrac{2}{3}ac}\cdot\sqrt{b^2+d^2+\tfrac{2}{3}bd}.$$

5. Show that the polynomial $x^n+x^{n-1}+x^{n-2}+...+x^2+x-1$ is irreducible over $\mathbb{Q}$.

6. There are N boys and N girls at the school. Each girl is acquainted with precisely K boys and each boy is acquainted with precisely K girls (the number K is the same for all boys and girls). For any two girls there are precisely C boys they are both acquainted with. Prove that for any two boys there are precisely C girls they are both acquainted with.

7. The set $\mathbb{L}=\mathbb{R}^2\times[0,\frac{1}{100}]\subset\mathbb{R}^3$ is colored in 4 colors. Prove that there exist two points in this set which are of the same color, and the distance between them is 1.

Good luck!

(אפשר לקחת שאלון איתכם בסוף המבחן)