First stage of Israeli students competition, 2016.

1. Compute $\det\begin{pmatrix} 5 & 1 & & & & & & \\ 1 & & 1 & & & & & \\ & 1 & 7 & 1 & & & & \\ & & 1 & & 1 & & & \\ & & & 1 & 7 & 1 & & \\ & & & & 1 & & 1 & \\ & & & & & 1 & 6 \end{pmatrix}$.

Answer. -25.

Solution. The determinant can be computed as a sum of products over all permutations. If "5" in the upper left corner participates in a permutation which gives a nonzero product, then we can't take any other element in first row or column, but we must take something both from second row and second column, which means that the cells $(2,3)$ and $(3,2)$ participate (we number rows and columns starting with 1), but then we can't take more numbers from second and third row or column, which leaves only cell $(4,5)$ in fourth row and only $(5,4)$ in the fourth column. The remaining two cells must be just in the two last rows and two last columns, so they are $(6,7)$ and $(7,6)$. The numbers in all these cells are 1, but the permutation is odd (three transpositions), so minus sign should be added, so the only contribution of permutation containing upper-left corner is -5.

Similarly, the contribution of the only permutation containing lower-right corner is -6, and it comes from different permutation.

Consider now permutations containing one of the "7" numbers, for instance in the cell $(3,3)$. It disallows 4 other nonzero cells, so in each of the lines 2, 4 and also rows 2,4 a unique choice remains: cells $(1,2), (2,1), (4,5), (5,4)$ must all be taken, and then from two last rows and two last columns we are forced to take the cells $(6,7)$ and $(7,6)$. So, the contribution is -7 and it comes from just one permutation, not using any other diagonal elements.

For similar reason, there's just one contribution of -7 from a permutation containing the cell $(5,5)$ and none of the other diagonal elements.

Now consider permutations not using diagonal elements, which is

$$\det\begin{pmatrix} 1 & & & & & & \\ 1 & & 1 & & & & \\ & 1 & & 1 & & & \\ & & 1 & & 1 & & \\ & & & 1 & & 1 & \\ & & & & 1 & & 1 \\ & & & & & 1 & \end{pmatrix}$$

This is zero, which is easy to see from chessboard coloring. The cell (i, j) of a matrix is called "black" if $i + j$ is even, and "white" if $i + j$ is odd.

Lemma. Any 7×7 matrix, all nonzero entries of which are white, has zero determinant.

Using this lemma we can complete the computation: permutations containing diagonal (black) cells contribute $-(5+7+7+6) = -25$, and others contribute zero. It still remains to prove the lemma.

Proof of the lemma. Each white cell either belongs to one of the three even rows, or to one of the three even columns. Therefore, the matrix of such type is sum of two matrices of rank at most 3. Therefore, the rank of the matrix is at most 6. Hence the matrix is degenerate.

2. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define a sequence of functions:

$f_n(x) = \int_0^x f_{n-1}(x)\,dt$. Compute $\lim_{n\to\infty} f_n(1000000)$.

Answer. 0.

Solution. We can choose a positive number M such that $|f(x)| \le M$ for each $x \in [0,1000000]$ (since f is continuous). Then

$$|f_2(x)| \le \int_0^x |f_1(t)|\,dt \le Mx \text{ for each } x \in [0,1000000],$$

$$\left|f_3(x)\right| \le \int_0^x M \cdot x \cdot dt \le M\,\frac{x^2}{2} \text{ for each } x \in [0,1000000],$$

$$\left|f_4(x)\right| \le \int_0^x M \cdot \frac{x^2}{2} \cdot dt \le M\,\frac{x^3}{3!} \text{ for each } x \in [0,1000000],$$

and so on, and therefore

$$\left|f_{n+1}(x)\right| \le M \cdot \frac{x^n}{n!} \text{ for each } x \in [0,1000000],$$

Take $n \ge 2000000$, and denote $C = M \cdot \dfrac{1000000^{1000000}}{1000000!}$.

Then $\left|f_n(100000)\right| \le C \cdot \left(\frac{1}{2}\right)^{n-1000000}$. Hence $f_n(1000000) \underset{n\to\infty}{\to} 0$.

3. Let V be the set of all vertices of a regular polygon. Might it be possible to find a subset $S \subset V$ so that the following two conditions are both satisfied:

(a) $|S| \ge \dfrac{99}{100} \cdot |V|$,

(b) union of any 10 rotations of S doesn't cover V ?

Answer. Yes, but a large number of vertices might be required.

First solution. Let us take a 10000^{10}-gon. Its vertices will be numbered in natural cyclic order, not with decimal numeration, but in base 10000 (meaning we use 10000 different digits instead of ten), by numbers of length 10.

The set S will consist of all numbers, not having at 0 or 1 as one of its digits.

Then $\dfrac{|S|}{|V|} = \left(1-\frac{2}{10000}\right)^{10} \ge \dfrac{99}{100}$, for instance by Bernoulli inequality:

$$\left(1-\frac{2}{10000}\right)^{10} \ge 1-10\cdot\frac{2}{10000} = 1-\frac{2}{1000} \ge 1-\frac{1}{100}.$$

A rotation of S is like adding a number to all elements. If digit at position k of the added number is a, then none of the numbers in the rotated set will have $a+1 \pmod{10000}$ as a digit at position k. Therefore, if $S_1, S_2, ..., S_{10}$ are ten rotations of S, we can take some digits $d_1, ..., d_{10}$ such that neither element of S_i has d_i at i'th position, so the number $\overline{d_1 d_2 ... d_{10}}$ is not in $S_1 \cup S_2 \cup ... \cup S_n$.

Solution. Consider n-gon. Let set S be chosen randomly. There are n^{10} ways to consider n different rotation of S. The probability of having a hole at some specific place is $0.01^{10}=10^{-20}$. The expectation of number of holes in the union of ten given rotations of S is $n\cdot 10^{-20}$.

4. Let $P(x)=2x^2-1$, and let $Q(x)=P\left(P\left(P(x)\right)\right)$. Let $R(x)$ be a polynomial of degree 8 such that $R(0)=1$, and $-1\le R(x)\le 1$ for each $x\in[-1,1]$. Prove that $Q(x)\le R(x)$ for each $x\in\left[-\frac{1}{10},\frac{1}{10}\right]$.

Solution. When $x\in\left[0,\frac{1}{10}\right]$, then $P(x)\in[-1,-0.98]$, and

$$P\left(P(x)\right)\in\left[2\left(1-\tfrac{2}{100}\right)^2-1,1\right]=\left[1-\tfrac{8}{100}+\tfrac{8}{10000},1\right]\subset\left[1-\tfrac{1}{10},1\right]=[0.9,1]$$

Therefore $Q(x)=P\left(P\left(P(x)\right)\right)\in\left[2\cdot 0.9^2-1,1\right]=[2\cdot 0.81-1,1]\subset[0.6,1]$.

So, $Q(x)$ is positive on $\left[0,\frac{1}{10}\right]$, and even $\left[-\frac{1}{10},\frac{1}{10}\right]$ since it is even.

While x goes from -1 to 1, $P(x)$ goes monotonically from 1 to -1 and back again monotonically from -1 to 1. So, $P\left(P(x)\right)$ goes from 1 to -1 then to 1 then to -1 and back to 1 (and monotonic on 4 subintervals). Similarly, $Q(x)=P\left(P\left(P(x)\right)\right)$ on $[-1,1]$ travels back and forth 8 times, starting from 1 and going monotonically once to -1 and back to 1, four times.

Polynomial $R(x)$ on interval $[-1,1]$ produces values in $[-1,1]$, so the graph of R intersects each of 8 monotonic segments in the graph of Q. It may happen that the graph of R meets two of the segments at their common endpoint. In these case, both R and Q are tangent to the same horizontal line at that common point. Either way, $R-Q$ has 8 roots counting with multiplicities (which is maximal allowed number, since the degree of polynomial is at most 8. Another way to say it that within the square $[-1,1]^2$, graphs of R and Q intersect at least 8 times, if tangency is counted as double intersection.

We are required to prove that $Q(x)\le R(x)$ for each $x\in\left[-\frac{1}{10},\frac{1}{10}\right]$. Assume the opposite: $Q(x_0)>R(x_0)$, where $x_0\in[-1,1]$. $x_0\neq 0$, since $R(0)=1$.

WLOG, $x_0 > 0$ (otherwise we could substitute $-x$ instead of x). The graph of R has to intersect the graph of Q at four monotonic segments to the right of x_0, at three leftmost monotonic segments, and it also is tangent to the graph at point $(0,1)$, so the number of intersections counted with multiplicities is at least 9. Therefore $R-Q$ has 9 roots counting with multiplicities, but it is a polynomial of degree 8, so it is identically zero. But $Q(x_0) > R(x_0)$, which is a contradiction.

5. Let ABC be a triangle a Euclidean plane, X a point in the same plane, and M the centroid (נקודת מפגש התיכונים) of the triangle. Show that

$$\frac{\mathrm{AX}^3}{\mathrm{AB}\cdot\mathrm{AC}} + \frac{\mathrm{BX}^3}{\mathrm{BA}\cdot\mathrm{BC}} + \frac{\mathrm{CX}^3}{\mathrm{CA}\cdot\mathrm{CB}} \geq 3\cdot\mathrm{MX}.$$

Solution. Let a,b,c be complex numbers representing points A, B, C in the complex plane, where coordinated are chosen so that X is 0, the origin.
The following identity is easily verified:

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=(a+b+c)(a-b)(a-c)(b-c)$$

We shall leave the computation to the reader. By triangle inequality

$$|a^3(b-c)|+|b^3(c-a)|+|c^3(a-b)|=|(a+b+c)(a-b)(a-c)(b-c)|$$

$$\frac{|a|^3}{|a-b|\cdot|a-c|}+\frac{|b|^3}{|b-a|\cdot|b-c|}+\frac{|c|^3}{|c-a|\cdot|c-b|}=|a+b+c|$$

This is precisely what we were required to prove, if you recall that $\frac{a+b+c}{3}$ represents M.

6. An open unit disc (in Euclidean plane) is covered by open regular triangles which are contained in the disc, but may overlap (in other words, each point inside the disc is inside one of the triangles, and each inside one of the triangles is inside the disc). Is it possible for the sum of sides of all triangles to be finite?

Answer. No.
Solution. There are two cases: the set of triangles might be countable or uncountable.

If the set is uncountable the sum of sides is infinite anyway. Indeed, let S_k be the set of triangles whose side is greater than $\frac{1}{k}$ but not greater than $\frac{1}{k-1}$ (where k is a natural number). If at least one among the sets S_k is infinite, then sum of the length even in S_k is infinite. If all are finite, their union is countable.

So, it enough to deal with the countable (or finite) case. There is uncountable set of directions, so we can rotate the picture in such a way, that neither of the triangles has horizontal side.

Each horizontal chord of the circle is covered by triangles, each triangle cuts an interval on such horizontal chords. Only countable number of endpoints of horizontal chords might be corners of triangle, so all horizontal chords except countable number intersect infinite number of triangles. Therefore almost all horizontal chords intersect infinite amount of triangles.

Integral of number of intersections of horizontal chords with triangles is precisely the total length of all projections of the sides of all triangles to the y-axis. We see that that integral is infinite, so the sum of all sides is infinite.