First stage of Israeli students competition, 2016.

Please try to write your solutions in English.

Duration: 4 hours

1. Compute $\det\begin{pmatrix} 5 & 1 & & & & & & \\ 1 & & 1 & & & & & \\ & 1 & 7 & 1 & & & & \\ & & 1 & & 1 & & & \\ & & & 1 & 7 & 1 & & \\ & & & & 1 & & 1 & \\ & & & & & 1 & 6 \end{pmatrix}$. The empty places are zeroes.

2. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define a sequence of functions: $f_n(x) = \int_0^x f_{n-1}(t)\,dt$.

Compute $\lim_{n\to\infty} f_n(1000000)$.

3. Does there exist a regular polygon, such that the set of its vertices V has a subset $S \subset V$, which satisfies the both following conditions:

(a) $|S| \geq \frac{99}{100} \cdot |V|$,

(b) the union of any 10 rotations of S doesn't cover V ?

4. Let $P(x) = 2x^2 - 1$, and let $Q(x) = P(P(P(x)))$. Let $R(x)$ be a polynomial of degree 8 such that $R(0) = 1$, and $-1 \leq R(x) \leq 1$ for each $x \in [-1,1]$. Prove that $Q(x) \leq R(x)$ for each $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$.

5. Let ABC be a triangle a Euclidean plane, X a point in the same plane, and M the centroid (נקודת מפגש התיכונים) of the triangle. Show that

$$\frac{AX^3}{AB \cdot AC} + \frac{BX^3}{BA \cdot BC} + \frac{CX^3}{CA \cdot CB} \geq 3 \cdot MX .$$

6. An open unit disc (in a Euclidean plane) is covered by open equilateral triangles (משולשים משוכללים) which are contained in the disc, but may overlap (in other words, each point inside the disc is inside one of the triangles, and each point inside one of the triangles is inside the disc). Is it possible for the sum of sides of all triangles to be finite?

Good luck!

(אפשר לקחת שאלון איתכם בסוף המבחן)