Second Stage of Israeli Olympiad for University Students.

1. On a plane N lines in **general position** are chosen. General position means, that no two lines are parallel, no three lines are concurrent, and neither three intersection points belong to the same unchosen line.
An additional line will be called good, if it doesn't pass through any intersection points of chosen lines. Additional lines will be considered equivalent, if one can be obtained from the other by continuous motion, such that all intermediate lines are good as well. Find the number of equivalence classes of good lines.

Answer. $\frac{n^4+2n^3-21n^2+10n+8}{8}$

Solution. We shall use projective duality. Consider central projection of the plane *P* where all the lines lift to sphere *S*. Each line will be projected to a *big circle*, i.e. circle of maximal radius. To each big circle one can associate a point - like pole associated to equator. If we glue together pairs of opposite points on sphere we get a *projective plane*.

Concurrent lines correspond to collinear points and vice versa. Using duality, we transform the problem to following one:

n points in general position on projective plane are given. General position *means that no 3 of them are collinear (and not any two of them are collinear together with the point, corresponding to infinite line (which has no projection on the plane P)), but we don't use this condition. Consider lines joining pairs of these points. Suppose no 3 of them are concurrent apart to the given points. In how many ways one can add one extra point non-collinear with any two others, if two positions are* equivalent *if one can be obtained to another by continuous motion such that in all intermediate positions no 3 points are collinear?*

This problem can be reformulated yet again in the following way:

n points in general position on projective plane are given. General position *means that no 3 of them are collinear. Consider lines joining pairs of these points. Suppose no 3 of them are concurrent apart from the given points. On how many regions they divide projective plane?*

Let us count. We have $m=\frac{n(n-1)}{2}$ lines joining these points. If they are in general position, they will divide ordinary plane on $\frac{m(m+1)}{2}+1$ part, $2m$ of them are infinite, $\frac{m(m-3)}{2}+1$ are finite. On the projective plane opposite infinite parts are glued, and we get *m* infinite and $\frac{m(m-3)}{2}+1$ finite parts, $\frac{m(m-1)}{2}+1$ parts in total.

However, our lines are not generic. Via each given point $n-1$ given lines are passing. If we slightly move them to a general position we get division of plane by $n-1$ lines with $\frac{(n-1)(n-4)}{2}+1$ bounded parts which will collapse when we return these lines to their initial place. Hence in each of *n* points we lose $\frac{(n-1)(n-4)}{2}+1$ parts, totally $n\left(\frac{(n-1)(n-4)}{2}+1\right)$.

So the number of parts *M* is equal

$$M=\frac{m(m-1)}{2}+1-n\left(\frac{(n-1)(n-4)}{2}+1\right)=$$

$$=\frac{\frac{n(n-1)}{2}\cdot\left(\frac{n(n-1)}{2}-1\right)}{2}-\frac{n(n-1)(n-4)}{2}-(n-1)=$$

$$=\frac{n-1}{8}\left(n^2(n-1)-2n+4n(n-4)-8\right)=$$

$$=\frac{n-1}{8}\left(n^3-n^2-2n+4n^2-16n-8\right)=\frac{n-1}{8}\left(n^3+3n^2-18n-8\right)$$

It is easy to see that the second term is indecomposable over $\mathbb{Q}$

2. Evaluate the integral $\int_0^\infty\left(e^{-2x}-e^{-x}\right)\frac{dx}{x}$.

Solution. Let's consider the more general problem,

$$I(a)=\int_0^\infty \left(e^{-ax}-e^{-x}\right)\frac{dx}{x}.$$

By differentiating we get

$$\frac{dI}{da}=\int_0^\infty -e^{-ax}dx.$$

The integral in the right hand side is absolutely convergent; therefore it can be integrated to obtain the original expression, hence differentiating inside the integral was justified.

$$\frac{dI}{da}=\int_0^\infty -e^{-ax}dx=\frac{e^{-ax}}{a}\Bigg|_0^\infty =0-\frac{1}{a}=0.$$

It is clear even to a hedgehog that $I(1)=0$. Hence

$$I(a)=\int_1^a \frac{dI}{da}\cdot da=-\int_1^a \frac{da}{a}=-\ln(a)$$

(we omit | | here, since for $a<0$ the integral is not defined anyway).

Hence $\int_0^\infty \left(e^{-2x}-e^{-x}\right)\frac{dx}{x}=I(2)=-\ln 2$.

3. Let $a,b,c,d,v\in\mathbb{R}^3$. Show that
$\langle[a,b],v\rangle\cdot\langle[c,d],v\rangle-\langle[a,c],v\rangle\cdot\langle[b,d],v\rangle+\langle[a,d],v\rangle\cdot\langle[b,c],v\rangle=0$.
Here $\langle,\rangle$ is scalar product and $[,]$ is vector product.

Solution. Denote the expression in the left-hand side by $f(a,b,c,d)$.

We claim, that f is anti-symmetric, which means that

$$f(v_1,v_2,v_3,v_4)=\operatorname{sgn}\sigma\cdot f\left(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)},v_{\sigma(4)}\right).$$

for each permutation σ of 4 indexes. Each anti-symmetric multi-linear expressions in 4 vectors in $\mathbb{R}^3$ is identically zero. Indeed, we could decompose each vector as a linear combination of vectors of the standard

basis, and represent $f(v_1,v_2,v_3,v_4)$ as a sum of $f(e_\alpha,e_\beta,e_\gamma,e_\delta)$ with coefficients (where e_1,e_2,e_3 is the standard basis and $\alpha,\beta,\gamma,\delta$ are indexes) and in each summands at least one index is repeated among $\alpha,\beta,\gamma,\delta$. If two among the vectors substituted in the form coincide and the form is anti-symmetric, the value is zero.

It is clear that our f is multi-linear, since both vector and scalar products are multi-linear. So, it remains to show it is anti-symmetric. Each permutation can be composed of just three transpositions $(1,2)$, $(1,3)$ and $(1,4)$ (in some order, maybe some of the transpositions are used more than once or not used at all). Therefore it is enough to verify the formula for only these 3 transpositions:

$$\begin{aligned} f(b,a,c,d) &= \langle [b,a],v\rangle\cdot\langle [c,d],v\rangle - \langle [b,c],v\rangle\cdot\langle [a,d],v\rangle + \\ &+ \langle [b,d],v\rangle\cdot\langle [a,c],v\rangle = -\langle [b,a],v\rangle\cdot\langle [c,d],v\rangle + \\ &+ \langle [b,d],v\rangle\cdot\langle [a,c],v\rangle - \langle [b,c],v\rangle\cdot\langle [a,d],v\rangle = -f(a,b,c,d) \end{aligned}$$

$$\begin{aligned} f(c,b,a,d) &= \langle [c,b],v\rangle\cdot\langle [a,d],v\rangle - \langle [c,a],v\rangle\cdot\langle [b,d],v\rangle + \\ &+ \langle [c,d],v\rangle\cdot\langle [b,a],v\rangle = -\langle [b,c],v\rangle\cdot\langle [a,d],v\rangle + \\ &+ \langle [a,c],v\rangle\cdot\langle [b,d],v\rangle - \langle [c,d],v\rangle\cdot\langle [a,b],v\rangle = -f(a,b,c,d) \end{aligned}$$

$$\begin{aligned} f(d,b,c,a) &= \langle [d,b],v\rangle\cdot\langle [c,a],v\rangle - \langle [d,c],v\rangle\cdot\langle [b,a],v\rangle + \\ &+ \langle [d,a],v\rangle\cdot\langle [b,c],v\rangle = \langle [b,d],v\rangle\cdot\langle [a,c],v\rangle - \\ &- \langle [c,d],v\rangle\cdot\langle [a,b],v\rangle - \langle [a,d],v\rangle\cdot\langle [b,c],v\rangle = -f(a,b,c,d) \end{aligned}$$

This completes the verification.

4. Let $\alpha > 0$ be an irrational number, $\beta \in (0,1)$. Denote:

$$Q(m) = \min_{1\le n\le m}\{n\alpha\}$$

$$R(m) = \min_{1\le n\le m}\{\beta - n\alpha\}$$

(where *n* runs over integer numbers between 1 and *m*).
Prove that there is infinite number of values m, such that $Q(m) > R(m)$.

Solution. We have to prove that for any natural N there is $n > N$ such that $Q_n > R_n$. The case when $R_n = 0$ for some n is obvious, so we suppose that $R_n > 0$ for all n, i.e. $\beta \neq \{k\alpha\}\ \ \forall k \in \mathbb{N}$. Because the set $\{n\alpha\}_{n=0}^{\infty}$ is dense, there exist $n \in \mathbb{N}$ such that $\{\beta - n\alpha\} \leq \min(\beta, R_n)$. Among such n we shall choose minimal n_0. Then

$$\{\beta - n_0\alpha\} < \min(\beta, R_n) \leq \{\beta - n\alpha\},\ \forall n \in [1, n_0 - 1].$$

Note that $n_0 > N$. Otherwise by the definition of R_n we would have an inequality $R_n \leq \{\beta - n\alpha\}$. In that case if $R_n < Q_{n_0}$ then taking $n_0 = n$ we get just what we need.

Now we can suppose that $R_{n_0} \geq Q_{n_0} = \{m\alpha\}$ for some $m \in [1, n_0]$. Then

$$R_{m_0+m} \leq \{\beta - \{n_0 + m\}\alpha\} = R_{n_0} - Q_{n_0} \qquad (1)$$

This holds because of equality

$$\{\beta - (n_0 + m)\alpha\} = \{\{\beta - n_0\alpha\} - \{m\alpha\}\} = \{R_{n_0} - Q_{n_0}\} = R_{n_0} - Q_{n_0}$$

We also have $Q_{n_0+m} = Q_{n_0} = \{m\alpha\}$.

Indeed. Suppose the contrary, i.e. $\{k\alpha\} < Q_{n_0}$ for some $k \in [n_0 + 1, n_0 + m]$. Then $0 \leq n_0 + m - k < n_0$ and
$\{\beta - (n_0 + m - k)\alpha\} < \{\beta - (n_0 + m)\alpha\} + \{k\alpha\} = R_{n_0} - Q_{n_0} + \{k\alpha\} < R_{n_0}$

This contradicts to the choice of n_0.

Now let $n_1 = n_0 + m$.

If $R_{n_1} < Q_{n_1}$ we can put $n + n_1$ and we are done. Otherwise if $R_{n_1} \geq Q_{n_1}$ we can replace n_0 by n_1 in the equations (1) and (2) and proceed. Then put $n_2 = n_1 + m = n_0 + 2m$ etc.

Finally for some k we get $R_{n_k} < Q_{n_k} - Q_{n_0}$ because if $R_{n_k} \geq Q_{n_k}$ for any k then using (1) and (2) (n_k substituted instead of n_0) we get

$$Q_{n_0} = Q_{n_0+m} = Q_{n_0+2m} = \ldots = Q_{n_0+km} \leq R_{n_0} - kQ_{n_0}$$

But for sufficiently large k we have $R_{n_0} - k \cdot Q_{n_0}$. This provides final contradiction.

5. What is the maximal possible area of an ellipse, which is contained in the upper half of a unit circle?

In the space of ellipses (center, orientation, and lengths of the axes) the set of admissible ellipses is compact; therefore a maximal ellipse exists.

First, it is obvious that the maximal ellipse should touch the boundary of the semicircle in at some points, such that their convex hull would contain the center of the ellipse. Otherwise, the ellipse can be moved to a certain direction and then homothetically expanded.

Let us prove that the maximal ellipse touches the diameter which is the boundary of the semicircle in the center. If not, we can move the ellipse a little bit towards the center, so that it will be no longer tangent to the circle but still won't contain the center. Then the ellipse can be expanded once again, so that it still won't contain the center and won't touch the circle. Of course, the ellipse can protrude outside the given semicircle, but it is still contained in some semicircle, since by convexity some line through the center of the circle doesn't intersect the ellipse. Now we can rotate the expanded ellipse back into the semicircle.

Now, there is a tangency point on the left and on the right half of the arc.

Assume the points are different; we'll denote them A and B.

Denote by ℓ the equation of the line AB and by ℓ_A, ℓ_B the equation of tangents at A and B. It is easy to see that all quadrics, passing through A, B and tangent to the circle there, are of the form $\lambda \cdot \ell_A \cdot \ell_B + \mu \cdot \ell^2$, and hence both the circle and the ellipse are symmetric with respect to the perpendicular bisector of AB.

Since the ellipse passes through the center of the circle, then AB is horizontal. Choose the natural coordinate system (the origin in the center of the circle, y upwards). Then the tangency points are $(\pm\cos t, \sin t)$.

We shall compute the area of the ellipse as a function of t.

Consider the affine transformation $T:(x,y)\mapsto(\alpha x,y)$ that will turn this ellipse into a circle. Before the transformation, the ellipse is inscribed in the triangle whose vertices are $\left(-\frac{1}{\cos t},0\right)$, $\left(\frac{1}{\cos t},0\right)$, $\left(0,\frac{1}{\sin t}\right)$, and the points of tangency are $(\pm\cos t,\sin t)$ and $(0,0)$.

After the action of T the ellipse becomes the incircle of the triangle, and the computation is easy.

But we have to find α. The tangents from $(\cos t,\sin t)$ to the ellipse should become equal, therefore the distances from $\left(\frac{\alpha}{\cos t},0\right)$ to both $(\alpha\cos t,\sin t)$ and $(0,0)$ should become equal. Hence

$$\left(\alpha\left(\cos t-\tfrac{1}{\cos t}\right)\right)^2+\sin^2 t=\frac{\alpha^2}{\cos^2 t}$$

and therefore by a standard computation,

$$\alpha^2=\frac{\sin^2 t}{1+\sin^2 t}$$

Now, the radius of the incircle can be obtained from the formula $S=pr$, where S is the area of the triangle, p is one-half its perimeter, and r is the radius of the incircle. From here, the area of the incircle is given by πr^2, and the area of the original ellipse becomes $\pi r^2/\alpha$. We compute:

$$S=\frac{1}{2}\cdot\frac{1}{\sin t}\cdot\frac{2\alpha}{\cos t}=\frac{\alpha}{\cos t\cdot\sin t}\ \text{ and }\ p=\frac{\alpha}{\cos t}+\sqrt{\frac{\alpha^2}{\cos^2 t}+\frac{1}{\sin^2 t}}.$$

This gives the area of the original ellipse is as follows:

$$f(t)=\frac{\pi S^2}{p^2\alpha}=\ldots=\pi\cdot\frac{\sin t}{\left(1+\sin^2 t\right)^{3/2}}$$

Deriving and finding the maximum gives

$$f'(t)=\pi\cdot\frac{1-\sin^2 t}{Z},\ \text{where}\ Z>0.$$

where Z is positive. This gives $t=45^\circ$, and $f(t)=\frac{2\pi}{\sqrt{27}}$.

Note that we have neglected the case where the ellipse is tangent to the circle only in one point, at $(0,1)$. In that case we may thicken the ellipse sideways until it is double tangent at that point, and hence this becomes a limit case of $f(t)$ where $t\to 1$, which not the maximum of f.

6. Let p be an odd prime, and let $GL_2(\mathbb{F}_p)$ be the set of all invertible 2×2 matrices over the field with p elements. A partition of $GL_2(\mathbb{F}_p)$ will be called **nice** if every two matrices belonging to the same set commute. Determine the minimal number of sets in the nice partition.

Answer. p^2+p+1

First solution. Let $\alpha\in\mathbb{F}_p$ a non-square, and set

$$a=\begin{pmatrix}1&0\\0&2\end{pmatrix},\ b=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ c=\begin{pmatrix}0&1\\\alpha&0\end{pmatrix}.$$

Their centralizers are easy to compute:

$$A:=Z_{GL_2(\mathbb{F}_p)}(a)=\left\{\begin{pmatrix}\beta&0\\0&\gamma\end{pmatrix}\ \middle|\ \beta,\gamma\in\mathbb{F}_p^\times\right\}$$

$$B:=Z_{GL_2(\mathbb{F}_p)}(b)=\left\{\begin{pmatrix}\beta&\gamma\\0&\beta\end{pmatrix}\ \middle|\ \beta\in\mathbb{F}_p^\times,\gamma\in\mathbb{F}_p\right\}$$

$$C:=Z_{GL_2(\mathbb{F}_p)}(c)=\left\{\begin{pmatrix}\beta&\gamma\\\alpha\gamma&\beta\end{pmatrix}\ \middle|\ \beta,\gamma\in\mathbb{F}_p,\ (\beta,\gamma)\neq(0,0)\right\}$$

This can be verified by an explicit computation, but also can be deduced from linear algebra: when a linear operator L is represented as a block matrix with smallest possible blocks, and all blocks are essentially different, then centralizer consists of matrixes with the same blocks, and in each block of size k we have a polynomial of degree $k-1$ at most in a relevant block of L.

Subgroups A,B,C are **commutative** (since they consist of linear expressions in respectively a,b,c). Let

$$\mathbf{A} := \left\{ gag^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{B} := \left\{ gbg^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{C} := \left\{ gcg^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{X} := \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$$

As the centralizer of each element of $\mathbf{X}$ is commutative, the commuting relation on $\mathbf{X}$ is an equivalence relation.

We claim that every element of $GL_2\left(\mathbb{F}_p\right)$ commutes with at least one element of $\mathbf{X}$. It is known from linear algebra that every 2×2 matrix can be transformed by conjugation to one of the following three canonical forms: $\begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ \alpha\gamma & \beta \end{pmatrix}$.

As the centralizer of each element of $\mathbf{X}$ is commutative, this proves that the minimal size of a nice partition of $GL_2\left(\mathbb{F}_p\right)$ and of $\mathbf{X}$ is the same. Indeed, we can extend every nice partition of $\mathbf{X}$ to a nice partition of $GL_2\left(\mathbb{F}_p\right)$ by adding each element $y \in GL_2\left(\mathbb{F}_p\right) \setminus \mathbf{X}$ to a set in the partition which contains some $x \in \mathbf{X}$ which commutes with *y*.

It remains to determine the number of sets needed for a nice partition of $\mathbf{X}$. As the commuting relation is an equivalence on $\mathbf{X}$, this is the same as determining the number of equivalence classes in $\mathbf{X}$ modulo the commuting relation.

We claim that no element in $\mathbf{A}$ commutes with an element of $\mathbf{B}$, no element in $\mathbf{A}$ commutes with an element of $\mathbf{C}$ and no element in $\mathbf{C}$ commutes with an element of $\mathbf{B}$.

Indeed, every element which commutes with some gag^{-1} is diagonalizable over $\mathbb{F}_p$, while non-scalar matrices commuting with some gbg^{-1} are not diagonalizable and non-scalar matrices commuting with gcg^{-1} are diagonalizable over $\mathbb{F}_{p^2}$ but not over $\mathbb{F}_p$.

Therefore it's enough to count the equivalence classes in **A**, **B**, **C** separately.

Lemma 1. The number of equivalence classes in **A** is

$$\frac{\left(p^2-1\right)\left(p^2-p\right)}{2\left(p-1\right)^2}=\frac{p\left(p+1\right)}{2}.$$

Proof. The only conjugate of a in A is $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, since A is diagonal matrices, and diagonal elements are eigenvalues which are uniquely defined up to order. By symmetry, the size of each equivalence class in **A** is 2. So $\left|\mathbf{A}\right|=\left|GL_2\left(\mathbb{F}_p\right)/A\right|=\frac{\left|GL_2\left(\mathbb{F}_p\right)\right|}{\left|A\right|}=\frac{\left(p^2-1\right)\left(p^2-p\right)}{\left(p-1\right)^2}$. The last number should be divided by 2.

Lemma 2. The number of equivalence classes in **B** is

$$\frac{\left(p^2-1\right)\left(p^2-p\right)}{p\left(p-1\right)^2}=p+1.$$

Proof. The only conjugates of b in B are $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$, where $x\in\mathbb{F}_p^{\times}$, since every conjugate of b has eigenvalue 1 with multiplicity 2. By symmetry, the size of each equivalence class in **B** is $p-1$. As

$$\left|\mathbf{B}\right|=\left|GL_2\left(\mathbb{F}_p\right)/B\right|=\frac{\left|GL_2\left(\mathbb{F}_p\right)\right|}{\left|B\right|}=\frac{\left(p^2-1\right)\left(p^2-p\right)}{p\left(p-1\right)}$$

To complete the proof, we divide $\left|\mathbf{B}\right|$ by the size of equivalence classes, which is $p-1$.

Lemma 3. The number of equivalence classes in **C** is

$$\frac{\left(p^2-1\right)\left(p^2-p\right)}{2\left(p^2-1\right)}=\frac{p^2-p}{2}.$$

Proof. The only conjugates of b in B are $\begin{pmatrix} 0 & 1 \\ \pm\alpha & 0 \end{pmatrix}$. This is because the eigenvalues of a matrix $\begin{pmatrix} \beta & \gamma \\ \alpha\gamma & \beta \end{pmatrix}$ are $\beta \pm \gamma\sqrt{\alpha}$ and this equals $\pm\alpha$ iff $\beta = 0$ and $\gamma = \pm 1$. By symmetry, the size of each equivalence class in $\mathbf{C}$ is 2. So $|\mathbf{C}| = \left|GL_2\left(\mathbb{F}_p\right)/C\right| = \dfrac{\left|GL_2\left(\mathbb{F}_p\right)\right|}{|C|} = \dfrac{\left(p^2-1\right)\left(p^2-p\right)}{\left(p^2-1\right)}$, which should be divided by 2.

To summarize, the minimal size of a nice partition is

$$\frac{p^2+p}{2} + \left(p+1\right) + \frac{p^2-p}{2} = p^2+p+1.$$

Second solution. As we have seen in the beginning of the first solutions, centralizer of non-scalar 2×2 matrix A consists of all possible (non-singular) matrices of the form $\lambda I + \alpha A$.

Notice that $gl_2\left(\mathbb{F}_p\right)/I$ is a 3-dimensional linear space, having a non-singular matrix in every line through the origin. Hence we need a separate class of partition for each point of a projective plane over $\mathbb{F}_p$, therefore we need precisely p^2+p+1 classes.