First stage of Israeli students competition, 2014.

Duration: 4 hours

1. A real number a is given. Find the greatest possible number of elements in the following sets:

a. $\{\sin x \mid \sin 3x = a\}$. b. $\{\sin x \mid \sin 4x = a\}$.

2. Compute $\det\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 1 & 5 & 3 \\ 3 & 5 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{pmatrix}$.

3. A grasshopper lives on a unit interval. Each second, a grasshopper chooses one of the endpoints of the unit interval (by flipping a coin) and jumps two-thirds of the way towards that endpoint (so the distance from him to one of the endpoints becomes precisely three times smaller).
130 spiders, each of the size 0.0005, can choose any positions on the interval simultaneously and stay there. Is it possible for them to catch the grasshopper?

4. For any $\varepsilon > 0$, consider the locus of points, satisfying the inequalities

$$\begin{cases} x^2 + y^2 + z^2 \le 1 \\ \left|x^2 + y^2 - z^2\right| \le \varepsilon \end{cases}$$

The volume of that locus will be denoted V_ε.

Does the $\lim\limits_{\varepsilon \to 0} \dfrac{V_\varepsilon}{\varepsilon}$ exist?

5. For any polynomial p with real coefficients, let

$$S(p) = \{x \in \mathbb{R} \mid p(x) \in \mathbb{Z}\}.$$

Prove that if p, q are two polynomials, such that $S(p) = S(q)$, then either $p + q$ or $p - q$ is a constant.

6. On a bookshelf, there are N tomes of the Encyclopedia in random order. Each hour, a librarian takes a tome which stands not on its place, and puts it in its place. Show that the process will stop.

Good luck!