University Students Olympiad, Stage 2

Duration - 4 hours

1. At the beginning of the game, on each cells of the lower half of the chessboard 8×8 is there is a white piece; on each cell of the upper half there is a black piece. On each move, the player is allowed to exchange two pieces on the cells that have a common side. In what minimal number of moves will he be able to move all black pieces to the lower half of the chessboard?

Solution. Let us invent the following "energy" function: let's number all lines from below, 1 to 8, and sum all line numbers of white pieces. The energy at the beginning is $(1+2+3+4)\cdot 8$ and it is lowest possible value. The energy in the end is $(5+6+7+8)\cdot 8$ and it is highest possible.

The difference is $(4+4+4+4)\cdot 8 = 4\cdot 4\cdot 8 = 2^{2+2+3} = 2^7 = 128$.

By each move the energy either remains the same, or changes by 1. It is increased if the move is vertical and white goes up. Therefore the total number of moves is at least 128, and it in each move one white piece goes up it will be precise.

The strategy is, therefore, to perform only vertical moves in which white goes up and black goes down. In this case, we can promise that at each moment, there are precisely 4 pieces of each color in each column (since pieces never move to another column). So, unless in each column white pieces are the top 4, we shall be able to find a white piece below the black piece and to improve the situation by 1.

2. Numbers a,b,c,d are integer and each two of them are coprime. Consider two intervals in the plane: interval I whose endpoints are $(0,0)$, (a,b) and interval J whose endpoints are $(0,0),(c,d)$. Two points are called **similar**, if both coordinates of vector connecting them are integer. Let n be the number of pairs of points (P,Q) which are similar, and are internal points of I and J respectively. Compute n in terms of a,b,c,d.

Solution 1. Actually, n is the number of integer points in Minkowski difference I – J which are not on the boundary.

I – J is a parallelogram, its sides don't contain integer points; its vertices are integer. By Pick formula $n+\frac{4}{2}-1=S$ which is the area of the parallelogram. On the other hand, we know it is the absolute value of the determinant $|ad-bc|$. Hence $n+1=|ad-bc|$ the answer follows.

3. Compute the integral $\int_0^{\pi/2}\ln\left(4-\sin^2 x\right)dx$.

Solution. We shall use the method of parametric differentiation.

Let us consider $I(t)=\int_0^{\pi/2}\ln\left(t^2-\sin^2 x\right)dx$. We have to compute $I(2)$.

Let us compute $\frac{dI}{dt}=\frac{d}{dt}\int_0^{\pi/2}\ln\left(t^2-\sin^2 x\right)dx=\int_0^{\pi/2}\frac{2t}{t^2-\sin^2 x}dx$.

Differentiation inside the integral is allowed for $t>1$, because the function under the integral is bounded. So, if we take $\tan x=y$ then

$\sin^2 x=\frac{\tan^2 x}{1+\tan^2 x}$ and

$$dy=d\tan x=d\left(\frac{\sin x}{\cos x}\right)=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx=\left(1+\tan^2 x\right)dx$$

so, $\frac{dy}{1+y^2}=dx$ and

$$\frac{dI}{dt}=\int_0^{\pi/2}\frac{2t}{t^2-\sin^2 x}dx=\int_0^{\infty}\frac{2t}{t^2-\frac{y^2}{1+y^2}}\cdot\frac{dy}{1+y^2}dy=\int_0^{\infty}\frac{2t}{t^2+y^2t^2-y^2}dy=$$

$$=\int_0^{\infty}\frac{2t}{t^2+y^2t^2-y^2}dy=\int_0^{\infty}\frac{2t}{t^2+y^2\left(t^2-1\right)}dy=\frac{2}{t}\int_0^{\infty}\frac{1}{1+y^2\left(1-\frac{1}{t^2}\right)}dy=$$

$$=\frac{2}{t\sqrt{1-\frac{1}{t^2}}}\arctan\left(y\sqrt{1-\tfrac{1}{t^2}}\right)\Big|_0^{\infty}=\frac{2}{\sqrt{t^2-1}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{t^2-1}}$$

So $\frac{dI}{dt}=\frac{\pi}{\sqrt{t^2-1}}$.

Therefore $I(t)=\int\frac{\pi}{\sqrt{t^2-1}}dt$. If $t=\operatorname{ch}s$ then $dt=\operatorname{sh}s\cdot ds$ and

$\int\frac{1}{\sqrt{t^2-1}}dt=\int\frac{\operatorname{sh}s}{\operatorname{sh}s}ds=s+const$. Since $t=\frac{e^s+e^{-s}}{2}$ so $0=e^{2s}-2te^s+1$

therefore $e^s=t+\sqrt{t^2-1}$ and $s=\ln\left(t+\sqrt{t^2-1}\right)$.

So, $I(t)=\int\frac{\pi}{\sqrt{t^2-1}}dt=\pi\ln\left(t+\sqrt{t^2-1}\right)+const$.

To find the constant, let us look at large t.

$$I(t)-\pi\ln\left(t+\sqrt{t^2-1}\right)=\int_0^{\pi/2}\ln\left(t^2-\sin^2x\right)dx-\pi\ln\left(t+\sqrt{t^2-1}\right)=$$

$$=\int_0^{\pi/2}\ln\left(t^2-\sin^2x\right)dx-\tfrac{\pi}{2}\ln\left(t^2\right)+\pi\ln t-\pi\ln\left(t+\sqrt{t^2-1}\right)=$$

$$=\int_0^{\pi/2}\left(\ln\left(t^2-\sin^2x\right)-\ln t^2\right)\cdot dx-\pi\ln\frac{t+\sqrt{t^2-1}}{t}=$$

$$=\int_0^{\pi/2}\ln\left(1-\frac{\sin^2x}{t^2}\right)\cdot dx-\pi\ln\left(1+\sqrt{1-\tfrac{1}{t^2}}\right)\xrightarrow[t\to\infty]{}\int_0^{\pi/2}\ln 1\cdot dx-\pi\ln(2)=$$

$$=-\pi\ln 2$$

Since the difference is constant,

$$I(t)=\pi\ln\left(t+\sqrt{t^2-1}\right)-\pi\ln 2=\pi\ln\left(\frac{t+\sqrt{t^2-1}}{2}\right)$$

Now, we substitute $t=2$ and get the answer $I(2)=\pi\ln\left(1+\frac{\sqrt{3}}{2}\right)$.

4. a. Compute the determinant $\begin{vmatrix}a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2\\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2\\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2\end{vmatrix}$.

b. Formulate and prove similar claim for $n\times n$ matrices instead of 3×3.

Solution. Notice that:

$$\begin{pmatrix} a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2 \\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2 \\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2 \end{pmatrix} =$$

$$= \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x & y & z \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x^2 & y^2 & z^2 \end{pmatrix}$$

If we multiply from the left by $\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}^{-1}$, and from the right by

$\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}^{-1}$, we get the matrix

$$e_3 \cdot e_1^T + e_2 \cdot e_2^T + e_1 \cdot e_3^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, our matrix is product of 3 matrices: two Vandermonde matrices and one permutation matrix. The determinants of Vandermonde are known to be $(c-a)(c-b)(b-a)$ and $(z-y)(z-x)(y-x)$. The permutation is negative. Hence the answer:

$$-(c-a)(c-b)(b-a)(z-y)(z-x)(y-x).$$

b. The natural generalization is matrix depending on two sequences $a_1,...,a_n$ and $x_1,...,x_n$ whose elements are $\left(x_i^{n-1} + x_i^{n-2}a_j + ... + a_j^{n-1}\right)$.

The proof is the same as in 3-dimensional case.

The answer is: $(-1)^{\left[\frac{n}{2}\right]} \prod_{i<k} (x_k - x_i) \prod_{j<\ell} (a_\ell - a_j)$.

If we take polynomials $\left(x_i^m + x_i^{m-1}a_j + ... + a_j^m\right)$, and $m < n-1$, then the determinant is zero, since the matrix is a sum of $m-1$ matrixes of rank 1 (by the same argument as above), and the rank is $m-1$ at most. If we take $m \geq n$, I don't know the answer.

5. ABCDE is a non-planar pentagon in three-dimensional Euclidean space. All sides of ABCDE are of the same length. The angles A, B, C, D of the pentagon are right. Find the angle E (all possible values).

Answer. Either 60° or $\arccos\left(\frac{5}{14}\right)$.

Solution. WLOG, length of each side is 1 (if not, apply homothety).
The intervals EB, EC are $\sqrt{2}$. Therefore, the triangle BEC is given.

One option is that ABCD is a square, and ADE is a regular triangle, such that plane ABCD is orthogonal to the plane ADE. In this case, angle E is 60°, or $\frac{\pi}{3}$ radians. In this case, we can choose Cartesian coordinates in space, s. t. $A=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $B=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $C=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $D=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $E=\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}$.

In any other case, we can still assume that coordinates of B,C,E are the same (since the triangle BCE is unique up to symmetry), and ask whether there are other possible options for A, D. On one hand, A should be in distance 1 from both B and E. This condition specifies intersection of two unit spheres with centers at B and E, or a circle in the plane of orthogonal bisector of BE. If A belongs to this circle, the condition of right angle at A is automatically satisfied, since the sides of triangle ABE are 1, 1, and $\sqrt{2}$, hence Pythagoras theorem holds.
Still, there is another condition on A: the angle ABC should be right. This condition specifies a plane through B, orthogonal to BC. Hence A should be on intersection of plane with a circle which lies in another (non-parallel) plane. This can happen only in two places A_1 which is the same as the A we guessed and A_2 which is symmetric to it with respect to the plane BCE.
Similar things can be said about the location of D: there is D_1 that we have guessed and there is D_2 symmetric with respect to BCE. It is easy to check, that at these points all conditions are satisfied. Therefore, there are only 4 possible pentagons, and they are symmetric in pairs. Pentagon A_1BCD_1E we have discussed already, it leads to 60°, its symmetric image A_2BCD_2E gives the same answer. It remains to discuss A_2BCD_1E (or

A_1BCD_2E; they are symmetric so it is enough to discuss one of them).

The equation of plane through $B=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $C=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $E=\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}$ can be written as $p(x,y,z)=\frac{1}{\sqrt{7}}\left(\sqrt{3}\cdot y-2\cdot z\right)=0$.

Its unit normal vector is $N=\begin{pmatrix}0\\\frac{\sqrt{3}}{\sqrt{7}}\\-\frac{2}{\sqrt{7}}\end{pmatrix}$. Then $p(A_1)=N\cdot A_1=-2/\sqrt{7}$.

The symmetric point $A_2=A_1-2(N\cdot A_1)N=\begin{pmatrix}0\\1\\0\end{pmatrix}-2\frac{\sqrt{3}}{\sqrt{7}}\begin{pmatrix}0\\\frac{\sqrt{3}}{\sqrt{7}}\\-\frac{2}{\sqrt{7}}\end{pmatrix}=\begin{pmatrix}0\\\frac{1}{7}\\\frac{4\sqrt{3}}{7}\end{pmatrix}$.

The angle A_2ED_1 is between the vector $ED_1=\begin{pmatrix}-\frac{1}{2}\\0\\\frac{\sqrt{3}}{2}\end{pmatrix}$ the vector

$EA_2=\begin{pmatrix}0\\\frac{1}{7}\\\frac{4\sqrt{3}}{7}\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\-\frac{6}{7}\\\frac{\sqrt{3}}{14}\end{pmatrix}$. Both are unit vectors; hence the cosine of each angle is the scalar product: $\cos\phi=\frac{1}{4}+\frac{3}{28}=\frac{10}{28}=\frac{5}{14}$. Therefore the result is $\arccos\left(\frac{5}{14}\right)$.

6. Denote by $\mathbb{R}_+$ the set of all nonnegative reals. We are given a sequence of functions $f_m:\mathbb{R}_+^m\to\mathbb{R}_+$, satisfying the following properties:

(a) Symmetry: $f_m(x_1,...,x_m)=f_m\left(x_{\sigma(1)},...,x_{\sigma(m)}\right)$ for any permutation σ.

(b) Monotonicity: $f_m(x,x_2,...,x_m)>f_m(y,x_2,...,x_m)$ if $x>y$.

(c) Homogeneous of degree 1: $f_m(\lambda x_1,...,\lambda x_m)=\lambda f_m(x_1,...,x_n)$, $\forall\lambda\in\mathbb{R}_+$.

(d) For any $k<m$:

$$f_m(x_1,...,x_k,x_{k+1},...,x_m)=f_m\left(f_k(x_1,...,x_k),...,f_k(x_1,...,x_k),x_{k+1},...,x_m\right).$$

(e) $f_2(0,1)=\frac{1}{2}$.

Prove that $f_m(x_1, x_2, ..., x_m) = \frac{1}{m}(x_1 + ... + x_m)$.

Sketch of solution. The problem is hard, and it would take much time to reduce everything to the axioms. Therefore we shall skip the trivial steps. Denote $R_{k,m} = f_m(\underbrace{1,...,1}_{k \text{ times}}, \underbrace{0,...,0}_{m-k \text{ times}})$. Then it follows from axioms that

$$f_{m^s}(\underbrace{1,....,1}_{k^s \text{ times}},0,...,0) = f_{m^s}(\underbrace{R_{k,m},....,R_{k,m}}_{m\cdot k^{s-1} \text{ times}},0,...,0) =$$

$$= R_{k,m} \cdot f_{m^s}(\underbrace{1,....,1}_{m\cdot k^{s-1} \text{ times}},0,...,0) = \left(R_{k,m}\right)^2 \cdot f_{m^s}(\underbrace{1,....,1}_{m^2\cdot k^{s-2} \text{ times}},0,...,0) =$$

...

$$= \left(R_{k,m}\right)^s \cdot f_{m^s}(1,...,1) = \left(R_{k,m}\right)^s$$

In particular, $f_{2^n}(1,0,...,0) = \frac{1}{2^n}$ and $f_{2^n}(\underbrace{1,...,1}_{2^m},0,...0) = \frac{2^m}{2^n}$ by similar argument. If in $f_n(1,...,1,0,...,0)$ we shall reduce the number of ones and increase the number of zeroes (maybe changing the n) the result will be reduced; if we increase the number of ones and reduce the number of zeroes, then the result will be increased; these follow easily from monotonicity.

Therefore if $k^s > 2^p, m^s < 2^q$, then $R_{k,m}^s < 2^{p-q}$.

Similarly if $k^s < 2^{p'}, m^s > 2^{q'}$, then $R_{k,m}^s > 2^{p'-q'}$.

Let us logarithm the last two statements:

If $s\log_2 k > p,\ \ s\log_2 m < q$, then $\log R_{k,m} < \frac{p-q}{s}$.

If $s\log_2 k < p',\ \ s\log_2 m > q'$, then $\log R_{k,m} > \frac{p'-q'}{s}$.

Obviously, if $p = \lfloor s\log_2 k \rfloor, p' = \lceil s\log_2 k \rceil, q' = \lfloor s\log_2 m \rfloor, q = \lceil s\log_2 m \rceil$, then the difference between the two bounds of $\frac{p'-q'}{s} < \log R_{k,m} < \frac{p-q}{s}$ is no greater than $\frac{4}{s}$, and both are within distance at most $\frac{2}{s}$ from $\log_2 k - \log_2 m = \log_2\left(\frac{k}{m}\right)$. Therefore $R_{k,m} = \frac{k}{m}$.