University Students Olympiad, Stage 2

Duration - 4 hours

1. At the beginning of the game, on each cells of the lower half of the chessboard 8×8 is there is a white piece; on each cell of the upper half there is a black piece. On each move, the player is allowed to exchange two pieces on the cells that have a common side. In what minimal number of moves will he be able to move all black pieces to the lower half of the chessboard?

2. Numbers a,b,c,d are integer and each two of them are coprime. Consider two intervals in the plane: interval I whose endpoints are $(0,0)$, (a,b) and interval J whose endpoints are $(0,0),(c,d)$. Two points are called **similar**, if both coordinates of vector connecting them are integer. Let n be the number of pairs of points (P,Q) which are similar, and are internal points of I and J respectively. Compute n in terms of a,b,c,d.

3. Compute the integral $\int_0^{\pi/2} \ln\left(4-\sin^2 x\right)dx$.

4. a. Compute the determinant $\begin{vmatrix} a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2 \\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2 \\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2 \end{vmatrix}$.

b. Formulate and prove similar claim for $n\times n$ matrices instead of 3×3.

5. ABCDE is a non-planar pentagon in three-dimensional Euclidean space. All sides of ABCDE are of the same length. The angles A, B, C, D of the pentagon are right. Find the angle E (all possible values).

6. Denote by $\mathbb{R}_+$ the set of all nonnegative reals. We are given a sequence of functions $f_m:\mathbb{R}_+^m\to\mathbb{R}_+$, satisfying the following properties:

(a) Symmetry: $f_m(x_1,...,x_m)=f_m\left(x_{\sigma(1)},...,x_{\sigma(m)}\right)$ for any permutation σ.

(b) Monotonicity: $f_m(x,x_2,...,x_m)>f_m(y,x_2,...,x_m)$ if $x>y$.

(c) Homogeneous of degree 1: $f_m(\lambda x_1,...,\lambda x_m)=\lambda f_m(x_1,...,x_n)$, $\forall\lambda\in\mathbb{R}_+$.

(d) For any $k<m$:

$$f_m(x_1,...,x_k,x_{k+1},...,x_m)=f_m\left(f_k(x_1,...,x_k),...,f_k(x_1,...,x_k),x_{k+1},...,x_m\right).$$

(e) $f_2(0,1)=\frac{1}{2}$.

Prove that $f_m(x_1,x_2,...,x_m)=\frac{1}{m}(x_1+...+x_m)$ for each m.

Good luck!