IMC 2013 Team Preselection Exam

Duration: 4 hours

January 23, 2013

Question 1 (10 points)

Let A be an $n \times n$ matrix with real entries and non-zero determinant such that for each $v \in \mathbb{R}^n$, the vectors Av and v are orthogonal. Prove that for each $v \in \mathbb{R}^n$, the vectors A^2v and v do not form an acute angle.

Solution:

There are several approaches here. The simplest one is:

$$\begin{aligned} 0 &= \langle Av + v, A(Av + v)\rangle = \left\langle Av + v, A^2v + Av\right\rangle \\ &= \left\langle Av, A^2v\right\rangle + \langle Av, Av\rangle + \left\langle v, A^2v\right\rangle + \langle v, Av\rangle = 0 + \langle Av, Av\rangle + \left\langle v, A^2v\right\rangle + 0, \end{aligned}$$

so

$$\left\langle v, A^2v\right\rangle = -\langle Av, Av\rangle \leq 0.$$

Question 2 (10 points)

Compute $\arctan^{(2012)}(0)$. (Reminder: $f^{(n)}$ means f derived n times.)

Solution:

We note that $\arctan'(x) = \frac{1}{1+x^2}$. However, it is well known that for $|x| < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots - x^2011 + x^2012 - x^2013 + \ldots,$$

and the sum converges absolutely in this range. Therefore,

$$\left(\frac{1}{1+x^2}\right)^{(2012)} = 2012! - \frac{2013!}{1!}x + \frac{2014!}{2!}x^2 - \ldots.$$

As a result,

$$\arctan^{(2012)}(0) = \left(\frac{1}{1+x^2}\right)^{(2012)}\Bigg|_{x=0} = 2012!.$$

Question 3 (10 points)

An $n \times n$ table consists initially of zeroes. At each step, it is allowed to choose a 2×2 subs-quare and revert all numbers in it: all zeroes are replaced by ones, and all ones are replaced by zeroes. How many possible tables can be created this way?

Solution:

We note that the order at which we invert the sub-squares does not matter, and that inverting the same sub-square twice does nothing. So, for each of the $(n-1)^2$ different sub-squares, we need to decide whether to invert it or not, giving us $2^{(n-1)^2}$ possibilites. Let us show that all of these possibilities are distinct.

Indeed, suppose that we inverted some set S_1 of sub-squares, and then inverted some set S_2 of sub-squares, and obtained the same result, with $S_1 \neq S_2$. So, obviously this means that inverting S_1 and then inverting S_2 brings the table back to zero (because every cell is inverted either an even number of times in both of them, or an odd number of times in both of them). However, according to the above, this is the same as inverting all 2×2 sub-squares in $(S_1 \cup S_2) - (S_1 \cap S_2)$, which is not empty because $S_1 \neq S_2$.

So, we have inverted some non-empty set S of sub-squares and returned to a board of 0-s. But, let us look at the sub-squares in S whose top edge is highest (there may be more than one). Among those, look at the one whose left edge is leftmost. In that sub-square, look at the top-left corner. But that corner is contained in exactly one sub-square (because all other sub-squares are either below it or to its right). So it was inverted only once, and hence has the value 1. A contradiction!

Question 4 (10 points)

A regular tetrahedron has only integer vertices. Show that its edge length divided by $\sqrt{2}$ is an integer number.

Solution:

Let the length of the side of the tetrahedron be ℓ. Let $0, u, v, w$ be the vertices of the tetrahedron. Then $|u| = |v| = \ell$, and becuase the angle between u and v (as vectors) is 60°, we see that

$$\frac{\ell^2}{2} = |u| \cdot |v| \cdot \cos(60^\circ) = u \cdot v \in \mathbb{Z},$$

since $u, v \in \mathbb{Z}^2$.

However, it is well known that the volume V of the tetrahedron is equal to $V = \frac{\ell^3}{6\sqrt{2}}$. In addition, it is also known that the volume of the tetrahedron is $\frac{1}{6}$ of the the volume of the paralelipiped spanned by u, v, w:

$$V = \pm\frac{1}{6}\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \in \frac{1}{6}\mathbb{Z}.$$

Thus, we have $\frac{\ell^3}{6\sqrt{2}} \in \frac{1}{6}\mathbb{Z}$, or $\ell^3/\sqrt{2} \in \mathbb{Z}$. So,

$$\frac{\ell}{\sqrt{2}} = \frac{\ell^3/\sqrt{2}}{2 \cdot \ell^2/2} \in \mathbb{Q}.$$

But this means that $\left(\frac{\ell}{\sqrt{2}}\right)^2 = \ell^2/2 \in \mathbb{Z}$, with $\frac{\ell}{\sqrt{2}} \in \mathbb{Q}$, which is well known to imply that $\frac{\ell}{\sqrt{2}} \in \mathbb{Z}$.

Question 5 (10 points)

Let A be an infinite subset of $\mathbb{N}$. Prove that there exists a number $\alpha > 100$ such that $B = \{\lfloor \alpha^n \rfloor \mid n \in \mathbb{N}\}$ satisfies: $A \cap B$ is an infinite set.

Solution:

Let A be an infinite subset of $\mathbb{N}$. We will inductively define numbers β_j such that the set B defined by each one of them will contain more and more elements of A, and such that they converge to a limit.

We have two main observations: one is that we can always shift β_j slightly without damaging anything, and the second is that if we raise β_j to a high enough power, then that small shift can allow it to "hit" any sufficiently large element of A.

Formally, suppose that we have found a number $\beta_j > 100$, and numbers $a_1, \ldots, a_j \in A$, $n_1, \ldots, n_j \in \mathbb{N}$, such that $a_k \leq \beta_j^{n_k} < a_k + \frac{1}{2}$ for all $1 \leq k \leq j$.

Then there is some $\epsilon > 0$ such that $\beta_j^{n_k}(1+\epsilon)^{n_k} < a_k + \frac{1}{2}$ for all $1 \leq k \leq j$ (this is a finite number of right-open constraints). Choose $n_{j+1} \in \mathbb{N}$ so huge that

$$(1+\epsilon)^{n_{j+1}} > \beta_j,$$

and $n_{j+1} > n_j$. Then there is some $a_{j+1} \in A$ such that $a_{j+1} > \beta_j^{n_{j+1}}$. Increasing n_{j+1} if necessary, we may assume that

$$\beta_j^{n_{j+1}+1} \geq a_{j+1} > \beta_j^{n_{j+1}}.$$

So, choose $\beta_{j+1} = a_{j+1}^{1/n_{j+1}}$. So, $\beta_j < \beta_{j+1} \leq \beta_j^{1/n_{j+1}}\beta_j < (1+\epsilon)\beta_j$. Hence:

$$a_k \leq \beta_{j+1}^{n_k} < ((1+\epsilon)\beta_j)^{n_k} < a_k + \frac{1}{2}$$

for all $1 \leq k \leq j+1$ (note that $a_{j+1} > \beta_j^{n_{j+1}} > \beta_j^{n_j} \geq a_j$, so $a_{j+1} > a_j$).

We repeat the above inductively (finding $100 < \beta_1$ such that $\beta_1^{n_1} = a_1$ with $n_1 = 1$ and $a_1 \in A$ is obvious). So, we have an infinite sequence of numbers, $100 < \beta_1 \leq \beta_2 \leq \ldots$, $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$, $a_1 < a_2 < a_3 < \cdots \in A$, such that

$$a_k \leq \beta_j^{n_k} < a_k + \frac{1}{2}$$

for all $1 \leq k \leq j$. In particular, $\beta_j^{n_1} < a_1 + \frac{1}{2}$ for all j, so β_j is a bounded increasing sequence. Let α be the limit. Then:

$$a_k \leq \alpha^{n_k} \leq a_k + \frac{1}{2}$$

for all k, so in particular $\lfloor \alpha^{n_k} \rfloor = a_k$ for all k.