First stage of Israeli students competition, 2013.

Duration: 4 hours

1. Let A be an $n\times n$ matrix with real entries and nonzero determinant, such that for each $v\in\mathbb{R}^n$, vectors Av, v are orthogonal. Prove, that for each $v\in\mathbb{R}^n$, vectors A^2v, v don't form an acute angle.

2. Compute $\arctan^{(2013)}(0)$. (Reminder: $f^{(n)}$ means: f derived n times).

3. A table $n\times n$ consists initially of zeroes. At each steps, it is allowed to choose a sub-square 2×2 and revert all numbers in it: all zeroes are replaces by ones, and all ones are replaced by zeroes. How many possible tables can be created in that way?

4. A regular tetrahedron has only integer vertices. Show that its edge length divided by $\sqrt{2}$ is an integer number.

5. Let A be an infinite subset of $\mathbb{N}$. Prove that there exists a number $\alpha>100$, such that $\mathrm{B}=\left\{\lfloor\alpha^n\rfloor \;\middle|\; n\in\mathbb{N}\right\}$ satisfies: $\mathrm{A}\cap\mathrm{B}$ is an infinite set.

Good luck!