Second stage of Israeli students competition, 2012.

Duration – 4 hours.

1. Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is positive definite.

2. A graph of a continuous function $f:[0,1]\to[0,1]$ is a broken line consisting of 10 intervals. The graph of the function $f\left(f\left(x\right)\right)$ is a broken line consisting of n intervals.
(a) What is the maximal possible value of n?
(b) What is the minimal possible value of n?

3. Let $p\left(x\right)$ be a non-constant polynomial with integer coefficients. Prove that there exists a natural number n, such that $p\left(n\right)$ has at least 5772 distinct prime divisors.

4. Let $K\subseteq\mathbb{R}^2$ be a convex shape, symmetric with respect to the origin. Suppose that $\iint_K dist\left(\vec{x},\partial K\right)d^2\vec{x}>2$, where $dist\left(\vec{x},\partial K\right)$ is the distance from x to the boundary of K. Prove that K contains at least 3 integer points.

5. Consider words consisting of zeroes and ones. Several words of length exactly 30 are considered **obscene**. A word is called **patient** if it is of infinite length and does not contain obscene subwords. Can it happen that a patient word exists, and every patient word is periodic of period greater than 10^9?

6. Let A, B be invertible matrices with integer entries, such that $\det A=1$.
(a) Can we claim that $B^{-1}AB$ has integer entries?
(b) Can we claim that there is a number n, such that $\left(B^{-1}AB\right)^n$ has integer entries?

Good luck!