IMC 2012 Team Preselection Exam

Duration: 4 hours

Janurary 6, 2012

Question1 (10 points)
Calculate the indefinite integral:

$$\int \frac{x^2}{(\cos x + x \sin x)^2} dx$$

Solution:

We use the following well-known trick. Let $x = \tan y$. Then:

$$\begin{aligned} \frac{1}{\sqrt{1+x^2}}(\cos x + x \sin x) &= \frac{1}{\sqrt{1+x^2}} \cos x + \frac{x}{\sqrt{1+x^2}} \sin x \\ &= \cos y \cos x - \sin y \sin x = \cos(y-x) \end{aligned}$$

Thus:

$$\begin{aligned} &\left(\frac{x}{\cos x + x \sin x}\right)^2 \\ &= \left(\frac{x/\sqrt{1+x^2}}{(\cos x + x \sin x)/\sqrt{1+x^2}}\right)^2 \\ &= \left(\frac{\sin y}{\cos(y-x)}\right)^2 \\ &= \frac{1-\cos^2 y}{\cos^2(y-x)} \end{aligned}$$

Therefore:

$$\begin{aligned} \int \frac{x^2}{(\cos x + x \sin x)^2} dx &= \int \frac{1-\cos^2 y}{\cos^2(y-x)} dx = \int \frac{dx - \cos^2 y dx}{\cos^2(y-x)} \\ &= \int \frac{dx - \cos^2 y d(\tan y)}{\cos^2(y-x)} = \int \frac{dx - dy}{\cos^2(y-x)} \\ &= \int \frac{d(x-y)}{\cos^2(x-y)} = \tan(x-y) \\ &= \frac{\tan x - \tan y}{1 + \tan y \tan x} = \frac{\tan x - x}{1 + x \tan x} \end{aligned}$$

Question2 (10 points)
Let $P(n)$ be a polynomial of degree 5772, with positive integer coefficients. Prove that

there is some integer n such that the decimal representation of $P(n)$ includes the digit 7 appearing 2011 times in a row.

Solution:

Note that if we take n and multiply it by a very large power of 10, then if we let $P(n) = \sum a_i n^i$, then the decimal representations of each monomial $a_i n^i$ are separated by a large number of zeroes. Hence it is sufficient to prove the question for any one monomial $P(n) = a_i n^i$. Specifically, it is sufficient to prove this for the top coefficient, $a_{5772} n^{5772}$, since $a_{5772} \neq 0$.

Now replace n with $n+1$. Then according to the same argument, it is sufficient to prove the statement for the linear coefficient, $P(n) = 5772 a_{5772} n$. Let $b = 5772 a_{5772}$. Then we must only consider $P(n) = bn$ with $b \neq 0$. Now we choose powers of 2 and 5 such that $2^x 5^y b = 10^z c$, where c is disjoint with 10. Then we substitute $n \mapsto 2^x 5^y n$ and since nothing changes if we divide $P(n)$ by a power of 10, we are reduced to the case $P(n) = cn$, where c and 10 are disjoint.

Then c is invertible modulu any power of 10, specifically, 10^{2011}. Therefore, we are left with $P(n) = n$, which is trivial.

Question3 (10 points)

Calculate the limit:

$$\lim_{n\to\infty} \left(n \left(\frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \cdots - \frac{1}{4n} \right) \right)$$

Solution:

Note that the expression in the limit is equal to:

$$\begin{aligned}
n \sum_{m=n}^{m=2n-1} \frac{1}{(2m+1)(2m+2)} &= n \sum_{m=0}^{m=n-1} \frac{1}{(2m+2n+1)(2m+2n+2)} \\
&= \sum_{m=0}^{m=n-1} \frac{1}{(2\frac{m}{n}+2+\frac{1}{n})(2\frac{m}{n}+2+\frac{2}{n})} \frac{1}{n} \\
&= O(\frac{1}{n}) + \sum_{m=0}^{m=n-1} \frac{1}{(2\frac{m}{n}+2)(2\frac{m}{n}+2)} \frac{1}{n} \\
&= O(\frac{1}{n}) + \sum_{m=0}^{m=n-1} \frac{1}{4} \frac{1}{(x_m+1)^2} (x_{m+1} - x_m)
\end{aligned}$$

Where $x_m = m/n$. Thus, by the definition of the integral:

$$\ldots \rightarrow \int_0^1 \frac{1}{4} \frac{1}{(x+1)^2} dx = \frac{1}{8}$$

Question4 (10 points)

For any m points $P_1 \ldots P_m \in \mathbb{R}^n$, let $A = (a_{ij})$ be the matrix defined by $a_{ij} = \|P_i P_j\|^2$. For a given n, what is the largest possible value of m for which A is invertible?

Solution:

We calim that the answer is $m = n + 2$. We do this by showing that the rank of A is no more than $m + 2$, and then showing that this rank is achieveable.

Indeed, let $\vec{u} = \underbrace{(1, \ldots, 1)}_{m}$, $\vec{w} = (P_1^2, \ldots, P_m^2)$ and for all $1 \leq i \leq n$, $\vec{v}_i = ((P_1)_i, \ldots (P_m)_i)$, where $(P_k)_i$ is the i-th component of P_k. Then we will show that $\vec{u}$, $\vec{w}$ and $\vec{v}_i$ span the rows of A. Indeed, since $a_{ij} = P_i^2 + P_j^2 - 2P_iP_j$ then the k-th row of A is: $P_k^2\vec{u} + \vec{w} - 2\sum_i (P_k)_i\vec{v}_i$, which is a linear combination of the above vectors. Thus the rank of A is no more than $n + 2$.

To show that this is indeed achieveable, let P_i for $1 \leq i \leq n+1$ be the vertices of the regular simplex in n dimensions, and P_{n+2} its center. We want to calculate A for this configuration, so we will give an explicit construction for this. Add another dimension, so that we are in $\mathbb{R}^{n+1}$. Then consider the $n+1$ unit vectors together with the vector $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. Then they are all on one hyperplane (defined by the equation $x + y + z + \cdots = 1$), and form the vertices and center of the regular simplex in that hyperplane (which has dimension n). Hence we can consider them to be vectors in $\mathbb{R}^n$.

Thus, it is easy to see that the distance between any two vertices of the simplex is $\sqrt{2}$, and that the distance to its center is $\sqrt{\frac{n}{n+1}}$. Thus A is:

$$A = \begin{pmatrix} 0 & 2 & \cdots & 2 & \frac{n}{n+1} \\ 2 & 0 & \cdots & 2 & \frac{n}{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 0 & \frac{n}{n+1} \\ \frac{n}{n+1} & \frac{n}{n+1} & \cdots & \frac{n}{n+1} & 0 \end{pmatrix}$$

And it is trivial to see that this matrix has maximal rank. Indeed, take the sum of the top $n + 1$ rows, and substract an appropriate multiple of the last row, to give us the vector $(0, \ldots 0, 1)$. Thus it is sufficient to show that the rows of the following matrix are independent:

$$\begin{pmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 0 \end{pmatrix}$$

Which is a well known fact.

Question5 (10 points)

A word w is called *square-free* iff it has no subword repeating twice in a row. Can you compose a square-free word using the 22 letters of the hebrew alpha-bet such that for any letter that we add on the left, the resulting word will no longer be square-free? If so, what is the minimal length of such a series?

Solution:

The answer is yes. In general, suppose that we have an alpha-bet of n letters $a_1, \ldots, a_n$. Define $w_1 = a_1$, and $w_i = w_{i-1}a_iw_{i-1}$. Then by induction, it is easy to see that w_i is square-free (if there is a subword repeating twice then it cannot contain a_i, thus w_i is not square-free) and also that for all i, a_iw_i contains a square. Thus, w_n is square-free, and

since it begins with w_i for all i, then the word $a_i w_n$ contains a square. Thus we have an example, of length $2^n - 1$.

Now we will prove that this is optimal. Let w be a word as in the question. Now let w_i be the shortest word such that $a_i w$ starts with $(a_i w_i)^2$ (the square word must always contain the first letter, otherwise w is not square-free). So for all i, w starts with $w_i a_i w_i$. Now suppose that w_i begins with $w_j a_j w_j$. Then we will say that $j \prec i$. Also, it is obvious that for each $i \neq j$, w_i starts with w_j or vice versa. Hence, assume that w_{i+1} starts with w_i for all i. We will show that for any $i < j$, $i \prec j$. From there, we will be done by induction. So suppose the contrary, that is, there is some i such that $i - 1 \not\prec i$.

Then suppose $w_i = w_{i-1} a_{i-1} u$, where u might be empty. Then we have $w_i a_i w_i = w_{i-1} a_{i-1} u a_i w_i$ begins with $w_{i-1} a_{i-1} w_{i-1}$. Then since the second w_{i-1} is not contained in the first w_i (otherwise $i - 1 \prec i$), we have $w_{i-1} = u a_i v$, where v might be empty. Then $w_i = w_{i-1} a_{i-1} u = u a_i v a_{i-1} u$.

But then we have that w starts with

$$\begin{aligned} w_i a_i w_i &= (u a_i v a_{i-1} u) a_i (u a_i v a_{i-1} u) \\ &= u a_i v a_{i-1} (u a_i)(u a_i) v a_{i-1} u \\ &= u a_i v a_{i-1} (u a_i)^2 v a_{i-1} u \end{aligned}$$

Therefore, w is not square-free, a constradiction!