Targil 1.

Linear algebra

1. Denote by $\{ f_n \}$ the Fibonacci sequence: $f_0 = 0,\ f_1 = 1,\ f_{n+2} = f_n + f_{n+1}$.
Prove that $f_{n+k} f_{n-k} - f_n^2 = \pm f_k^2$.
(What is the sign in this formula?)

2. Every entry of an N×N matrix is ±1. What is the maximal possible determinant of this matrix
a. for N = 4?
b. for N = 8?

3. Find the maximal volume of a simplex which is contained in a unit cube of dimension N for
a. N = 3.
b. N = 7.
Remark. A simplex is a higher-dimensional generalization of a two-dimensional triangle and a three-dimensional tetrahedron; in other words, it is a convex hull of N+1 points which are not in one hyperplane (which happens to be a minimal number of points which might be not in one hyperplane).

4. We are given nonzero n×n matrices $A_1, A_2, \ldots, A_k$.
a. Over an infinite field, prove that it is possible to find a matrix B such that the matrix $A_1BA_2B \ldots BA_k$ is nonzero.
b. Is this true for a finite field?

5. A linear transformation $f : \mathbb{Q}^n \to \mathbb{Q}^n$ satisfies $f\left(f\left(f\left(v\right)\right)\right) = 2v$ for any vector v.
Describe all the possible values of n.