First stage of Israeli students competition, 2011.

1. Find all possible values of $\lim_{x\to\infty} x^{(\ln x)^\lambda}$ for real λ.

Solution. $\lim_{x\to\infty} x^{(\ln x)^\lambda} = \lim_{x\to\infty} e^{(\ln x)(\ln x)^\lambda} = \lim_{x\to\infty} e^{(\ln x)^{\lambda+1}} = e^{\lim_{x\to\infty}(\ln x)^{\lambda+1}} = e^{\lim_{y\to\infty} y^{\lambda+1}}$,

where $y = \ln x$. The power $\lim_{y\to\infty} y^{\lambda+1}$ can be:

0 if $\lambda+1<0$,
1 if $\lambda+1=0$,
$+\infty$ if $\lambda+1>0$.
Therefore, the original limit can be either 1, e, or $+\infty$.

2. Is it possible to draw a pentagon with integer coordinates of vertices and equal sides?

Answer: no.
Solution. Consider the pentagon of that kind with minimal side. It is defined if such pentagons exist, because the distance between integer points is a root of an integer number (and each non-empty subset of nonnegative integers has minimal element). The number under the root is even if the endpoints are of the same color, and odd if they are of opposite color (here we use the standard chess coloring, point are black if the sum of its coordinates is even, and white if the sum of its coordinates is odd). So, if the length of each side is a square root of an odd number then each two adjacent vertices have different chess colors, which is not possible since 5 is odd. Therefore the length of each side must be even, and all the vertexes must have the same chess color. So, if we rotate the picture around one of the vertices by 45º and reduce it $\sqrt{2}$ times, we shall get another polygon with integer vertexes and equal sides, but this time the sides are shorter. This contradicts the assumption.

3. Compute $1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\frac{1}{10}+\frac{1}{11}-\frac{2}{12}+\ldots$.

First solution: Denote $S_n = 1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\ldots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}$. It is enough to compute $\lim_{n\to\infty} S_n$ (and to prove that it exists). Indeed, S_n is 3n'th partial sum, and the $3n$+1st and $3n$+2nd partial sums are close to it (the distance is less than $1/n$).

$$S_n = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3n}\right)-3\left(\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{3n}\right)=$$
$$=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3n}\right)-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)=\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)=$$
$$=\frac{1}{n}\left(\frac{1}{(n+1)/n}+\frac{1}{(n+2)/n}+\frac{1}{(n+3)/n}+\ldots+\frac{1}{3}\right) \underset{\text{sum}}{\overset{\text{Riemann}}{=}} \int_1^3 \frac{dx}{x} = \ln x\Big|_1^3 = \ln 3$$

Second solution. Consider logarithmic Taylor series: $-\ln(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$

Substitute $\xi = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ (it satisfies $\xi^3 = 1$). The series $\xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots$ will converge by Dirichlet criterion (since $\xi+\xi^2+\xi^3+\ldots+\xi^n$ is bounded by 2, and $\frac{1}{n}\to 0$ monotonically). For $|x| < 1$ the series converge to $-\ln(1-x)$. The series and the complex continuation of $-\ln(1-x)$ are both continuous in the domain of convergence of the series, therefore $-\ln(1-\xi) = \xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots$.

In the complex continuation of ln, we have $\mathrm{Re}(\ln(z)) = \ln|z|$. Therefore

$$-\ln|1-\xi| = \mathrm{Re}\left(\xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots\right) = \mathrm{Re}\left(\xi+\frac{\overline{\xi}}{2}+\frac{1}{3}+\frac{\xi}{4}+\frac{\overline{\xi}}{5}+\frac{1}{6}+\ldots\right)=$$
$$=-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{6}+\ldots$$

Multiplying the right hand side by -2 will give us the series we want to compute. So, it is equal to the left hand side times -2, which is $2\ln|1-\xi| = \ln|1-\xi|^2 = \ln\left((1-\xi)\cdot\overline{(1-\xi)}\right) = \ln\left((1-\xi)\cdot(1-\overline{\xi})\right) = \ln(1-\xi-\overline{\xi}+1) = \ln 3$.

4. Michal and Ohad play a game in which Michal marks points and arcs in the plane, and Ohad assigns colors to the points. Michal makes the first move. In each of her moves, Michal marks one point, and she can also join it by arcs to some of the existing points, provided that the arcs do not intersect (except possibly in endpoints). Ohad, in turn, paints the last marked point in some color, which must be different than colors of endpoints connected to this point by an arc. Michal wins, if Ohad will use more than 5771 colors. Does Michal have a winning strategy?

Answer: Yes.
Solution. We consider the painting as a graph, where the vertices are the points, and the edges between the vertices are the arcs connecting them. Recall that a graph is called a *forest* if it contains no cycles, and the connected components of a forest are called *trees*.

Basic Lemma: Suppose the painting so far is a forest. Then any two points can be connected by a new arc legally. **Why?** Because the arcs do not form cycles, they cannot split the plane into several parts, meaning that the plane with the arcs removed is still one connected face. In particular any two points on it can be connected by an arc not intersecting those previously marked.

Corollary: Suppose the painting so far is a forest. Then after marking a new point, Michal can always choose one point from each tree in the painting, and connect all those points to the new point marked: Indeed, she may simply draw these arcs one after the other, as every arc only joins together two connected components, and does not form cycles – meaning the graph remains a forest after every step, and thus she can draw the next arc by the basic lemma.

Claim: For any k, Michal can construct a forest F_k with N_k vertices (where N_k is some finite number) which Ohad will be forced to paint with at least k different colors. Michal then wins the game by building a copy of F_{5772}.

Proof: By induction: Basis is k=1, where we may simply mark a new point, and it will be colored in one color and be a forest (so $N_1=1$). Next we assume that it is true for all numbers up to k, and prove it for k+1: To construct F_{k+1}, Michal should

first build a copy of $F_1, F_2, \ldots, F_k$. In F_1, there is at least one color used: Denote such one as c_1, and let v_1 be some point in F_1 with that color. In F_2 at least two colors are used, so there must be at least one used other than c_1, denote it c_2, and v_2 is some point in F_2 with that color. Continue in this fashion: For all values of m up to k, at least m colors are used in F_m, so in particular there must appear a color c_m which is different from $c_1,\ldots,c_{m-1}$, and a vertex v_m with that color. Now, as all vertices v_i are in different trees (as they are in disjoint forests), by the corollary above, Michal can mark a new point and connect it to all vertices v_i. Ohad must then paint the new point in a color different from all colors $c_1,\ldots,c_k$, and it is thus clear that with the new point there will be at least k+1 different colors in the graph. Furthermore, the new graph is also clearly a forest, so it is an example of F_{k+1}, with a clearly bounded number of vertices.

Remarks. It can be seen that the above construction actually yields a tree, not a forest. It is easy to verify that the above construction gives $N_k=2^{k-1}$. It is possible not to demand necessarily that F_k are forests, but to demand that on each step in the construction, the new point marked is accessible from the unbounded component of the plane, and stop as soon as you get the new color wanted.

5. An infinite sequence of positive real numbers satisfies:

$$\det\begin{pmatrix} 1 & a_i & 0 & a_i^2 \\ 2 & \sqrt{3}\cdot a_{i+1} & a_{i+1} & 2a_{i+1}^2 \\ 2 & a_{i+2} & \sqrt{3}\cdot a_{i+2} & 2a_{i+2}^2 \\ 1 & 0 & a_{i+3} & a_{i+3}^2 \end{pmatrix} = 0$$

Prove that it is periodic.

Solution. Let us rewrite the determinant as follows:

$$\det\begin{pmatrix} 1 & a_i & 0 & a_i^2 \\ 1 & \frac{\sqrt{3}}{2}\cdot a_{i+1} & \frac{1}{2}\cdot a_{i+1} & a_{i+1}^2 \\ 1 & \frac{1}{2}\cdot a_{i+2} & \frac{\sqrt{3}}{2}\cdot a_{i+2} & a_{i+2}^2 \\ 1 & 0 & a_{i+3} & a_{i+3}^2 \end{pmatrix} = 0$$

This actually means that the following four points

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_i \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} a_{i+1} \\ \frac{1}{2} a_{i+1} \end{pmatrix} \quad ; \quad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} a_{i+2} \\ \frac{\sqrt{3}}{2} a_{i+2} \end{pmatrix} \quad ; \quad \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ a_{i+3} \end{pmatrix}$$

belong to one curve which is described by an equation of the following type: $k + lx + my + n\left(x^2 + y^2\right) = 0$. Such equations can describe either a circle (if n is nonzero) or a line (if n is zero).

The length of the above four vectors are a_i, and each one is 30º counter-clockwise with respect to the previous. Therefore, if we define a sequence of vectors, such that the vector number i has argument $30°i$ and length is a_i.

Each 4 consequent points will be either on one circle or one straight line. Each three non-collinear points define a unique circle, therefore all points in a sequence will be either on one line or one circle.

If we will have a line, or a circle which doesn't contain the origin, then one of the first seven rays won't even intersect it. If the line or the circle goes through the origin, one of a_i will have to be zero, and it is given that they are positive. Therefore, the points in the sequence are intersections of the rays with some circle, which goes around the origin. This defines each point uniquely. Rays number i and i +12 coincide, therefore, $a_i = a_{i+12}$. So the sequence is periodic (and the period divides 12).