First stage of Israeli students' competition, solutions.

1. Compute: a. $\det\begin{pmatrix}1&4&9\\4&9&16\\9&16&25\end{pmatrix}$ b. $\det\begin{pmatrix}1&4&9&16\\4&9&16&25\\9&16&25&36\\16&25&36&49\end{pmatrix}$

Answers. a. -8 **b.** 0

Solution. a. Subtract second line from the third. After that, subtract first line from the second. We get:

$$\det\begin{pmatrix}1&4&9\\4&9&16\\9&16&25\end{pmatrix}=\det\begin{pmatrix}1&4&9\\4&9&16\\5&7&9\end{pmatrix}=\det\begin{pmatrix}1&4&9\\3&5&7\\5&7&9\end{pmatrix}$$

Now do the same with columns (subtract second from third, first from second):

$$\det\begin{pmatrix}1&4&9\\3&5&7\\5&7&9\end{pmatrix}=\det\begin{pmatrix}1&4&5\\3&5&2\\5&7&2\end{pmatrix}=\det\begin{pmatrix}1&3&5\\3&2&2\\5&2&2\end{pmatrix}$$

Now subtract second row from the third, and then second column from the third:

$$\det\begin{pmatrix}1&3&5\\3&2&2\\5&2&2\end{pmatrix}=\det\begin{pmatrix}1&3&5\\3&2&2\\2&0&0\end{pmatrix}=\det\begin{pmatrix}1&3&2\\3&2&0\\2&0&0\end{pmatrix}$$

Here we have only one permutation with nonzero product, which is the secondary diagonal. The product is 8, but the permutation is negative. So it is -8.

b. The determinant can be written as follows

$$\det\begin{pmatrix}f(1)&f(2)&f(3)&f(4)\\g(1)&g(2)&g(3)&g(4)\\h(1)&h(2)&h(3)&h(4)\\k(1)&k(2)&k(3)&k(4)\end{pmatrix}$$

where $f(x)=x^2, g(x)=(x+1)^2, h(x)=(x+2)^2, k(x)=(x+3)^2$.

Polynomials of degree 2 form 3-dimensional linear space, hence *f*, *g*, *h*, *k* are linearly dependent. Therefore the rows of matrix are linearly dependent. So the determinant is 0 (of course, we could compute it by subtracting the rows/columns, but it is always better to get the result without computation).

2. a. How many planes are required to cut all the edges of a cube?
b. How many planes are required to cut each edge of a cube twice?
Remark. Edges of a polytope (מקצועות של פאון) are the intervals which are the sides of its faces. We say that a plane cuts an interval if the plane contains precisely one internal point of that interval.

Answer. a. 3. b. 4
Solution. a. It is easy to build an example of 3 planes cutting all the edges of a cube: for instance, for each pair of parallel faces take a plane which is parallel to both and is between them. The tricky part is to show why 2 planes are not enough.

The nicest explanation I saw belongs to Dan Carmon and was invented during the competition. Suppose two planes *A*, *B* cut all edges. Consider the third plane *C* which is orthogonal to both planes. We can rotate the cube slightly, so that planes *A* and *B* will still cut the same edges but *C* won't be orthogonal to any faces of the cube. Project the picture orthogonally to the plane *C*. In the projection, the cube becomes a convex hexagon, and planes *A*, *B* become straight lines. These two straight lines should cut all 6 of the convex hexagon. But each line can cut only two of them. This yields a contradiction.

b. An intersection of a plane and a cube is a polygon with at most 6 sides. The reason is the following: each side is intersection of a plane and a face, and the cube has only 6 faces (maybe less, since the plane doesn't have to intersect all faces). To cut each of 12 edges 6 times we need at least 24 intersection points, so at least 24 angles in our intersection polygons; each plane contributes at most 6, so at least 4 planes are needed.

A perpendicular bisector plane to any diagonal of the cube (by diagonal of a cube we mean an interval connecting two opposite vertices) cuts precisely 6 edges at their midpoints. The cube has 4 diagonals; so we can have 4 such planes, the picture is symmetric, so each edge is cut by the same number of planes, so it is 2 (since the total number of intersection is $4\times6 = 24$). One could complain that the different planes intersect any edge in the same point, precisely in the middle. To fix this issue, it is enough to shift all 4 planes by very small, but different distances.

3. Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(x+y)-f(x-y)=2y\cdot f'(x) \qquad \forall x,y\in\mathbb{R}.$$

Solution. Derive by y:

$$f'(x+y)+f'(x-y)=2f'(x)$$

$$\frac{f'(x+y)+f'(x-y)}{2}=f'(x)$$

Therefore, for each two points on the graph, the middle point is also on the graph. Therefore, the interval connecting this two points and the graph of f' coincide on a dense set of points (the middle of the interval, the middle of subintervals formed by the midpoints, the middles of smaller subintervals formed by those points etc.). That fact, along with continuity of the function, implies that f' is linear on any interval, therefore it is linear. That means that f is quadratic function.
It is easy to see that any quadratic function satisfies the original equation (exercise to the reader).

4. The Department of Social Equality has 15 workers. In the beginning, each has a salary which is a positive integer number of NIS no greater than 10. Each year, the boss can raise the salaries of exactly 13 workers by 1 NIS simultaneously. Workers are immortal, they never quit or retire; new workers are never accepted. The boss wants to make the salaries of all workers in the Department equal.
a. Prove that it is possible.
b. In the worst case, how many years will it take?

Answer. 70.
Solution. First of all, raising the salary of 13 by 1 is the same as decreasing salaries of two by 1, at least to the people who think in abstract mathematical terms and are interested only in social equality and not in actual money. That way, we might eventually arrive to negative salaries, but so be it.
a. We can split all people except A into pairs and reduce salary of each pair. This is the same as raising salary to A by 1. Using operations of that kind, we can clearly arrive to the equality.
b. Assume that 13 workers have 10 NIS salary, one worker has 9 NIS salary, and the last one has 1 NIS salary. We reduce salaries of two workers a year. To have an equality, we have to come to a situation, in which all get no more than 1 NIS a year. However, it cannot be that all would get 1 NIS a year, even after a long time,

since the total of all salaries is even and it shall never be odd. So, to achieve social equality, we have to make all their salaries at most 0. Their total salay at the beginning is 140, in the end 0 at most, so at least 70 years are required in this case.

It remains to prove that it cannot be more than 70. Without loss of generality, we may assume that the last worker has salary 1 NIS in the beginning. If the total salary is odd, we might try to arrive to a situation in which the salary of all workers is 1; if the salary is even, we might try to arrive to a situation where the salary of all workers is 0. In the beginning, the largest possible total salary is 141 (all except the last worker get 10), the largest possible even total salary is 141, and the largest possible odd total salary is 140. If it is possible, it would take no more than 70 years to make all salaries to be the same number, which is 1 or 0.

If we would also be allowed to reduce salary to the same person twice during the same year (instead of reducing each time the salaries of two different workers) then it would obviously be possible. We can write a plan, how to arrive to social equality in at most 70 years, with two people in each year, but in some years the same person can be mentioned twice. We can assume this plan is at least for 50 years: if not, we can add 15 more years, while in those additional years the salary is reduced to each worker. Now we shall reorganize this plan so that nobody will be reduced twice in the same year.

Notice, that each worker starts with a salary at most 10, and arrives to the salary at least -9 (because bringing all people to salary -10 means total salary would be -150 and for that 75 years at least would be required). So salary of each worker will be reduced less than 20 times. Therefore, if a salary of some worker according to the plan is reduced twice in one year, it is possible to find a year in which his salary is not reduced (since that plan has 50 years at least). We shall swap one name between these two years, and the number of such bad years will be reduced. We can do it as long as bad years exist, so after a finite number of operations we shall have a plan with the same number of years in which bad years don't exist.

5. A function c over the set of natural numbers is defined as follows:

$c(n) =$ 0 if there are even number of ones in binary representation of n,

1 otherwise.

A positive integer number k is given.
Let $l(N)$ be the number of integers n from 0 to N, such that $c(k+n) \neq c(n)$.
Prove that $\lim\limits_{N\to\infty} \dfrac{l(N)}{N}$ exists and belongs to $\left[\dfrac{1}{3}, \dfrac{2}{3}\right]$.

Solution. Denote $p(k) = \lim\limits_{N\to\infty} \dfrac{l(N)}{N}$, where k is the number that was used to define function l.
First compute the limit for $k = 1$. The parity is switched if the binary number has in its end 0, or 011, or 01111, or 0111111, … and limit of density of these numbers exists and equals $p(1) = \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{32} + \ldots = \dfrac{1}{2} \Big/ \left(1 - \dfrac{1}{4}\right) = \dfrac{1}{2} \Big/ \dfrac{3}{4} = \dfrac{2}{3}$.
We shall prove the claim by induction. Suppose we have proved the claim for all numbers smaller then k, and now we prove it for given k.
If $k = 2m$ then adding k to the number is the same as adding m the number for which the last binary digit is erased. Since $p(m)$ exists and belongs to $\left[\dfrac{1}{3}, \dfrac{2}{3}\right]$, so does $p(k)$.
Now suppose $k = 2m + 1$. Comparing $c(k+n)$ to $c(n)$ splits into two cases: when $n = 2s$, then $c(k+n)$ differs from $c(n)$ if and only if $c(m+s)$ equals $c(s)$, because last binary digit is changed, and to the rest of the number m is added. Therefore the limit of probability that $c(k+n)$ differs from $c(n)$ exists and equals to $1 - p(m)$.
If $n = 2s + 1$, then the last digit is changed anyway, from 1 to 0, and we have a carry, and then we should actually add $m + 1$ to the number which is n after erasing the last digit. So for odd numbers limit density also exists and equals $1 - p(m + 1)$.
If we take both even and odd numbers, the limit also exists and equals the average (since from 1 to N there's almost equal number of evens and odds) which is $p(k) = \dfrac{(1 - p(m)) + (1 - p(m+1))}{2}$. Since both $1 - p(m)$ and $1 - p(m+1)$ belong to $\left[\dfrac{1}{3}, \dfrac{2}{3}\right]$ by induction, their average also belongs to the same interval.