Final stage of Israeli students competition, 2010.

Duration: 4.5 hours

1. Prove that there exists an integer n, such that $n^2 + 3$ is divisible by 7^{5770}.

Solution. There's nothing so special about this year, so we shall prove that for every natural m, we can find n such that $n^2 + 3$ is divisible by 7^m.
The base of induction is simple: $2^2 + 3$ is divisible by 7.
The step of induction: assume $k^2 + 3 = q \cdot 7^m$. Things are already right modulo 7^m, so let us search for n in the form $k + x \cdot 7^m$.
Then $n^2 + 3 = (k + x \cdot 7^m)^2 + 3 = q \cdot 7^m + 2kx7^m + x^2 7^{2m}$.
The last term in the sum us already divisible by 7^{m+1}, so all we need is to find q for which $(q + 2kx)7^m$ is divisible by 7^{m+1}. In other words, we need to solve

$$q + 2kx = 0 \pmod 7$$

It is possible, since k is not divisible by 7 (otherwise $k^2 + 3$ wouldn't be divisible by 7). QED.
Remark. This easy exercise can be generalized into so-called Hensel's lemma.

2. The union of 2010 people is divided into two non-empty fractions. Every fraction of more than one person is also divided into two nonempty fractions (recursively). Every person must pay a monthly contribution of one shekel for every fraction that contains him.

(a) What is the minimal possible total of all monthly contributions?
(b) What is the maximal possible total of all monthly contributions?

Answers. (a) 20100 + 1972 = 22072 (b) 2009·1006 = 2021054
Solution. We can form a binary tree: the root will be the union, and all nodes will be fractions, the sons of each set are the two fractions into which it is subdivided. Since every man is in a fraction of one person, people correspond to the leafs of that tree. Therefore the total of monthly contributions corresponds to the total of depths of all leafs (the root has depth zero, sons of vertex of depth k have depth k+1). By conditions, there are exactly 2010 leafs, and every non-leaf has exactly two sons. It is obvious that both minimum and maximum exist, since the number of configurations is finite.

(a) Consider arbitrary tree satisfying the conditions. Assume that leaf A has depth k and leaf B has depth greater than k+1. Father of leaf B is C, and it has depth at least k+1. Consider the following surgery: both sons of C are reattached to A. That way, A stops to be a leaf, but C becomes a leaf, so number of leafs is preserve. All relocated nodes of the tree reduce their depth. Therefore, the total monthly contribution is reduced.

Therefore, if the total monthly contribution is minimal, then there are only two possible depths for leafs: k and k+1.

Now consider the situation with minimal total contribution.

For each leaf of vertex at depth d its weight will be defined as 2^d. Sum of all the weight is obviously 1. If some of the weights are 1/4096 or less, than all the weights are at most 1/2048, so number of people is at least 2048 which is impossible.

Similarly, if some of the weights are 1/512 or more, than all weights are at least 1/1024, so we have at most 1024 people.

Hence only possible weights are 1/1024 and 1/2048.

Number of people with weight 1/1024 will be denoted by M. We can say that all 2010 people have basic weight 1/2048 but M of them have extra weight of 1/2048. Hence 2010/2048 + M/2048 = 1. In other words, M = 38.

These people pay 10 sheqels a month, other 2010 – 38 = 1972 pay 11 sheqels a month. We could say that in this minimal situation all 2010 people pay 10 sheqels, but 1972 people pay an extra sheqel, so the total is 20100 + 1972 = 22072.

(b) Again, take some configuration. Assume some node C have two sons, A and B. We claim that if the situation is maximal, one of them should be a leaf.

If not, both all sons of A and reattach then to some leaf D which is a descendant of B. The number of leafs is still the same, but depth of all nodes that were relocated (and the leafs among them) is increased, QED.

Hence in the maximal situation we have one leaf on depth one, one leaf on depth 2, … etc, and two leafs on depths 2009. Sum of depths of all leafs is

$1+2+3+\ldots+2009 + 2009 = 2009\cdot 2010/2 + 2009 = 2009(1005+1)=2021054$.

3. $\{a_n\}$ is a non-decreasing sequence of real numbers such that $a_1 = 1$, $a_2 = 2$, $a_{mn} = a_m a_n$. Find a_3 (all possible values).

Answer. $a_3 = 3$.

First solution. There is a possibility that satisfies all conditions $a_n = n$ for every n.
In this case $a_3 = 3$.
It remains to prove there are no other options.
Obviously $a_3 > a_2 > 0$.
If $2^k > 3^m$, then $2^k = (a_2)^k = a_{2^k} > a_{3^m} = (a_3)^m$.
But if $2^r < 3^s$, then $2^r = (a_2)^r = a_{2^r} < a_{3^s} = (a_3)^s$.
Equivalent way to say the same thing: if $\frac{k}{m} > \log_2 3$, then $\frac{k}{m} > \log_2 a_3$, but if $\frac{r}{s} < \log_2 3$, then $\frac{r}{s} < \log_2 a_3$.
In other words, there are no rational numbers between $\log_2 3$ and $\log_2 a_3$.
It means that the numbers $\log_2 3$ and $\log_2 a_3$ are equal (directly from definition of real numbers). But $\log_2 3 = \log_2 a_3$ yields $3 = a_3$. QED.

Second solution. Suppose $a_3 = a$. Then $a_3^{\,k} = a^k$.
If $2^m < 3^k < 2^{m+1}$, then also $2^m < a^k < 2^{m+1}$.
Hence $\frac{1}{2} < \left(\frac{3}{a}\right)^k < 2$ for all k. Therefore $\frac{3}{a} = 1$.

4. (**a**) A and B are symmetric positive definite $n \times n$ matrixes (with real coefficients). AB is symmetric. Prove that AB is positive definite.
(b) A, B, C are symmetric positive definite $n \times n$ matrixes (with real coefficients). ABC is symmetric. Prove that ABC is positive definite.

Reminders. Change of basis in matrix which has the geometrical meaning of linear transformation is $P \cdot A \cdot P^{-1}$. Change of basis in a matrix which has the geometrical meaning of bilinear form is $P \cdot A \cdot P^T$, where P is an invertible matrix. The latter keeps properties such as of being symmetric / positive definite, the former keeps spectrum (eigenvalues), and all things which are derived from them (trace, determinant, etc).
A well-known theorem claims that a symmetric matrix can be diagonalized by an orthogonal matrix, which means we can diagonalize it in both meanings simultaneously.

However, in the case of the bilinear form, we might want not to stop at orthogonal matrixes, but apply also stretching in different direction; then we shall get the theorem, saying that every bilinear form is in some basis represented by diagonal matrix, with only 0, 1, or -1 at each diagonal element (this theorem is named after English (Jewish) mathematician James Joseph Sylvester).

Solution. (a) First solution. We can choose some orthonormal basis, in which A is diagonal, in our case with positive numbers over the diagonal. Therefore, WLOG, A is a diagonal matrix with positive numbers d_i^2 on the diagonal (here we may and will assume $d_i > 0$), entries of B will be denoted b_{ij}. Let D be a diagonal matrix with numbers d_i on the diagonal.

Matrix $AB = D^2B$ is symmetric. Multiplying by D from the left is multiplying row number i by d_i. Therefore elements of matrix AB are $d_j^2 \cdot b_{ij}$. Since it is symmetric,

$$d_j^2 \cdot b_{ij} = d_i^2 \cdot b_{ji} = d_i^2 \cdot b_{ij}$$

It can be true in two cases: either $d_j^2 = d_i^2$ which means $d_j = d_i$ since they are positive, or $b_{ij} = 0$. In both cases,

$$d_j^2 \cdot b_{ij} = d_i \cdot b_{ij} \cdot d_j$$

therefore, AB = DBD. Since D is non-degenerate, it follows that AB is equivalent to B, hence it is also positive definite.

Second solution. $AB = (AB)^T = BA$. Both A and B are diagonalizable, and they commute, so they have a common eigenbasis. In that basis, both A and B are diagonal with positive numbers on the diagonal, and the statement is obvious.

(b) Let us diagonalize B by orthogonal matrix: $B = P \cdot G \cdot P^T$. Since G is diagonal and positive definite, we can write is as H^2, where H is diagonal matrix with square roots of corresponding G elements on the diagonal.

Therefore $B = P \cdot D \cdot D \cdot P^T = PD \cdot (PD)^T = Q \cdot Q^T$, where Q is an invertible matrix.

Therefore we can write

$$ABC = AQQ^TC = (Q^T)^{-1} \cdot Q^T \cdot A \cdot Q \cdot Q^T \cdot C \cdot Q \cdot Q^{-1}$$

Matrixes $A_1 = Q^T \cdot A \cdot Q$ and $B_1 = Q^T \cdot C \cdot Q$ are symmetric positive definite, since they are equivalent to A and C respectively.

$$Q^TABCQ = A_1B_1$$

It is a symmetric matrix, so it is positive definite by (a).

5. Compute $\int_0^{2\pi} e^{\cos x}\cos(\sin x)\,dx$.

Answer. 2π.

Solution.

$$\int_0^{2\pi} e^{\cos x}\cos(\sin x)\,dx = \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x}\cdot\left(\cos(\sin x)+i\cdot\sin(\sin x)\right)dx\right)=$$

$$=\mathrm{Re}\left(\int_0^{2\pi} e^{\cos x}e^{i\cdot\sin x}\,dx\right)=\mathrm{Re}\left(\int_0^{2\pi} e^{\cos x+i\cdot\sin x}\,dx\right)=\int_0^{2\pi}\mathrm{Re}\left(e^{e^{ix}}\right)dx$$

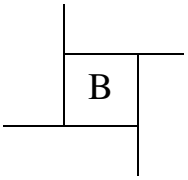
e^z is a holomorphic function, hence it's real part is harmonic. Therefore, average of this function over any circle equals value at the center. the unit circle e^{ix} is equal its value at 0. Here we have the integral over the unit circle e^{ix}, so we should take a value at center $e^0 = 1$ and multiply it by the length of the circle.

6. A box is subdivided into a finite number of cubes (more than one).
Prove that two of the cubes are of the same size.

Solution. Assume that all cubes are of different size. Consider cube A, which is the smallest cube among all cubes that touch the floor of the box.
The cubes that touch the floor of the box divide this floor into squares, and A gives the smallest square among those. The vertical walls of A are surrounded by either walls of the box, or by higher neighboring cubes. So, on the ceiling of A we have a "well".
The ceiling of A cannot be covered from above by one cube, since it would be same size as A. Therefore ceiling of A is covered by several cubes (which are strictly above A because of the walls of the "well").
Let B be the smallest among the cubes that stand on the ceiling of A.
B cannot be in the corner of the well, because one of its neighbors which stand on the ceiling of A would be not greater than B (there are at least to neighbors, one on the east/west wall, another on the north/south wall. If both are greater than B, they intersect.

B cannot be near the wall of the well for similar reason (one of its neighbors on ceiling of A in one of the three direction will be not greater than B, otherwise they intersect).

That leaves us with only one possible configuration of the neighbors of B on the ceiling of A, like the one in the picture. Since all 4 neighbors at the sides of B are taller than B, the ceiling of B is also surrounded by "walls".



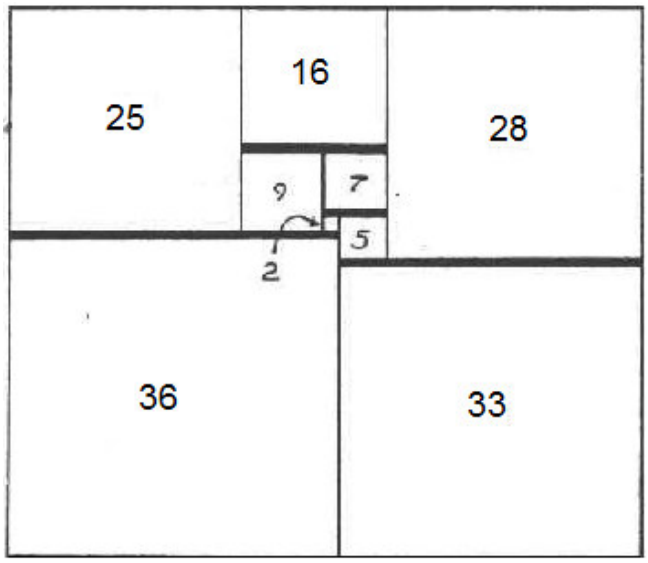
We may continue repeating this argument, and we shall get an infinite sequence of cubes, each one standing on the previous cube and strictly smaller than it and surrounded by "walls". This contradicts the assumption that there was a finite number of cubes.

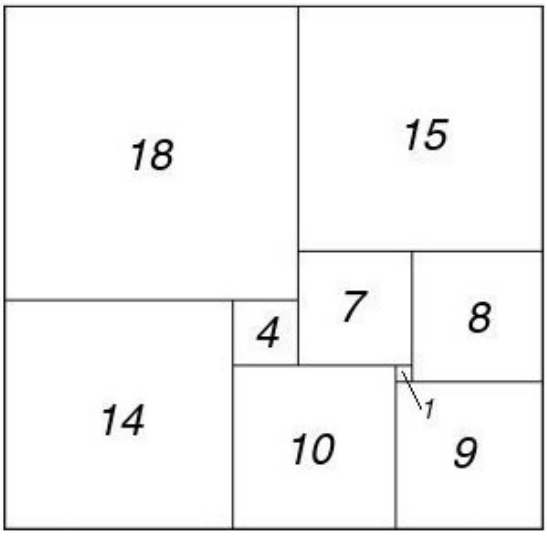
Remark. This statement was proved in a work of four Cambridge students, (Brooks, Smith, Stone and Tutte) which they did in late thirties (of twentieth century). This was the last small statement in their paper, which also included several exciting results such as:

A idea of how to relate this problem in dimension 2 to electricity(!).

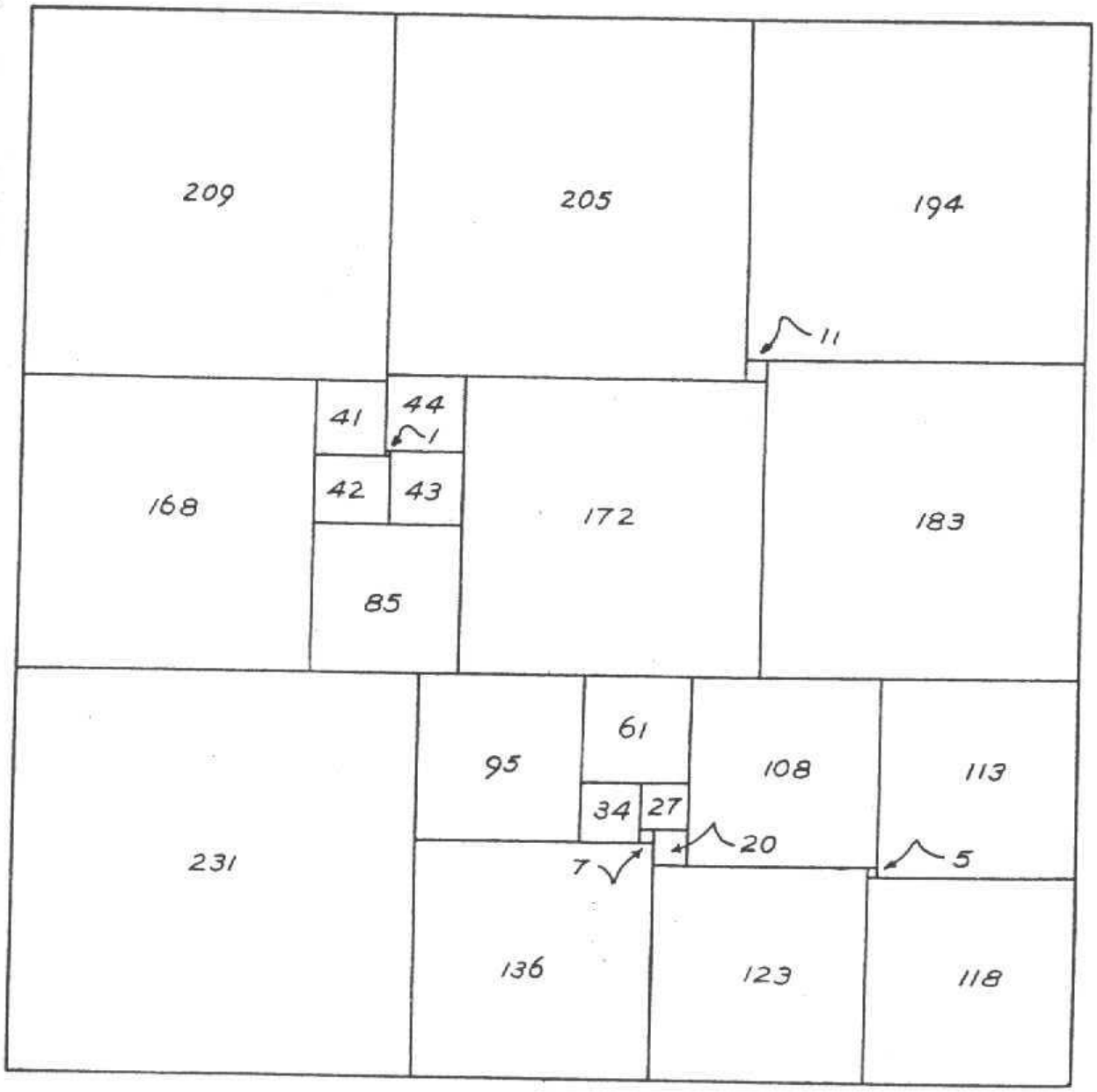
Using this idea, a method of constructing counterexamples in dimension 2 (that is, dividing rectangles into squares, such that all squares have different sides).

Both examples of rectangles divided into minimal number of squares (which is 9).





Example of a subdivision of a square into squares of different size (26 squares, which was a contradicts conjecture by a famous Russian mathematician N. Luzin).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