Final stage of Israeli students competition, 2010.

Duration: 4.5 hours

1. Prove that there exists an integer n, such that $n^2 + 3$ is divisible by 7^{5770}.

2. The union of 2010 people is divided into two non-empty fractions. Every fraction of more than one person is also divided into two nonempty fractions (recursively). Every person must pay a monthly contribution of one shekel for every fraction that contains him.

(a) What is the minimal possible total of all monthly contributions?

(b) What is the maximal possible total of all monthly contributions?

3. $\{a_n\}$ is a non-decreasing sequence of real numbers such that $a_1 = 1$, $a_2 = 2$, $a_{mn} = a_m a_n$. Find a_3 (all possible values).

4. **(a)** A and B are symmetric positive definite $n \times n$ matrixes (with real coefficients). AB is symmetric. Prove that AB is positive definite.

(b) A, B, C are symmetric positive definite $n \times n$ matrixes (with real coefficients). ABC is symmetric. Prove that ABC is positive definite.

5. Compute $\int_0^{2\pi} e^{\cos x} \cos(\sin x)\, dx$.

6. A box is subdivided into a finite number of cubes (more than one). Prove that two of the cubes are of the same size.

Good luck!