Second stage of Israeli students competition, 2010.

1. Let $K = \sum_{n \in \mathbb{N}} 2^{-n^2}$ (where $\mathbb{N}$ is a set of positive integers).

Is *K* rational or irrational?

Answer. *K* is irrational.

Solution. Write a binary fraction of *K*. It is not periodic, since it has longer and longer sequences of zeroes. Hence the number is irrational.

2. ABCD is a tetrahedron (not necessarily regular). Denote *a* = distance between the lines AB and CD, *b* = distance between the lines AC and BD, *c* = distance between the lines AD and BC. Prove that the volume can't be less than $\frac{abc}{3}$.

Solution. Move the tetrahedron so that the center of mass of A, B, C, D will be 0. In other words, A + B + C + D = 0.

Construct 4 points K = – A , L = – B , M = – C , N = – D.

$$AL + CN = L - A + N - C = -B - A - C - D = 0$$

So vectors AL and NC are equal, and ALCN is a parallelogram.

Similarly, vector AL coincides with vector MD and BK.

Therefore ALDM and MDKB and BKCN are parallelograms. Hence points A, B, C, D, K, L, M, N are the vertexes of parallelepiped.

The common perpendicular to AB and CD is orthogonal to two faces of the parallelepiped. Therefore *a* is the distance between faces AMBN and CKDL. Similarly, *b* and *c* are distances between other pairs of parallel faces.

Lemma. If *a*, *b*, *c* are distances between parallel faces of parallelepiped then the volume $V \geq abc$.

The parallelepiped consists of tetrahedrons ABCD, ABCN, BCDK, ACDL, ABDM. It is easy to see that the last 4 have volume *V*/6 each, so volume of ABCD is *V*/3. It remains to prove the lemma.

Proof of lemma. Volume = altitude of parallelepiped on AMBN times altitude of face AMBN on AM times AM. First factor is *a*, second is greater than *b*, third is greater than *c*.

3. We have a system of *L* lamps and *B* buttons. Each button has one of two states: "on" or "off". Each button is connected to several lamps. A lamp may be connected to more than one button. Pressing a button toggles all connected lamps to the opposite state.
(a) Prove that number of potential states of lamps is a power of 2.
(b) Suppose that for every subset S of lamps there is a button switching state of odd subset of S. Prove that all lamps can be switched off.

Solution. (a) Consider linear space over a field of 2 elements of dimension *L*. Each button specifies a vector in that space (coordinated 1 if the lamp is connected, coordinate 0 otherwise). Combination of several buttons corresponds to the sum of their vectors. Sums of given vectors form a linear subspace. Since it is still linear space over the field of two elements, number of possible changes that can come from combinations of buttons is a power of two.
(b) All possible changes that might be applied to the system form a linear subspace in the space of all conceivable changes. If this subspace would not be everything, it would lie in a hyperplane. That means all possible changes would have zero scalar product with a given nonzero vector. That means there's a subset of lamps, such that every possible change flips even number of lamps in that subset. This is specifically forbidden by the condition.

4. Compute $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

Solution. Denote $f(x)=\frac{x\sin x}{1+\cos^2 x}$.

$$\int_0^{\pi} f(x)dx = \frac{1}{2}\int_0^{\pi} f(x)+f(\pi-x)dx = \frac{1}{2}\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{-d(\cos x)}{1+\cos^2 x} =$$

$$= \frac{\pi}{2}\int_1^{-1}\frac{-dy}{1+y^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{dy}{1+y^2} = \frac{\pi}{2}\arctan\Big|_{-1}^{1} = \frac{\pi}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

5. A is a 2×2 matrix with integer coefficients. Absolute values of all entries of A are less than 10. Absolute values of all entries of A^{1000} are less than 10^9. Prove that they are actually less than 10^6.

Solution. First consider determinant. det $A = d$ which is an integer number.
det $A^{1000} = d^{1000}$
If $|d| \geq 2$, then det $A^{1000} \geq 2^{1000} \geq 1000^{100} = 10^{300}$ and that is definitely not a determinant of a matrix having absolute values of entries less than 10^9.
Therefore, the d is 1, 0, or -1.
We shall use the following lemma.

Lemma. Absolute values of eigenvalues of A are less than 1.2.

Proof of lemma. Assume A has an eigenvalue k such that $|k| \geq 1.2$ and it corresponds to an eigenvector v. Then $k^2 = 1.44 > \sqrt{2}$, hence $k^4 > 2$, and $k^{40} > 2^{10} > 1000$.
Therefore $k^{1000} > 1000^{25} = 10^{75}$. But $A^{1000}v = k^{1000}v$.
The vector is multiplied by a number, of absolute value greater than 10^{75}; it wouldn't happen if all entries of A^{1000} are less than 10^9. QED of lemma.

Therefore, absolute value of trace(A) can't be 3 or greater: then absolute value of one eigenvalue would be above 1.5 (since trace is the sum of eigenvalues) and that can't happen. Hence trace(A) can be 0, ±1, or ±2. Multiplying the whole matrix A by -1 doesn't change A^{1000}, so we may assume that trace(A) is nonnegative.

To summarize: WLOG det(A) is -1, 0 or 1 and trace(A) is 0, 1 or 2.
This gives us 9 possibilities. Let us check all of them:

a) det(A) = -1 and trace(A) = 2.

Eigenvalues satisfy $x^2 - 2x - 1 = 0$ which is $x^2 - 2x + 1 = 2$ so they are $\frac{1 \pm \sqrt{2}}{2}$. The greater eigenvalue is $\frac{1+\sqrt{2}}{2} > \frac{1+1.4}{2} = 1.2$, and this can't happen because of the lemma.

b) $\det(A) = 0$ and $\text{trace}(A) = 2$.

One eigenvalue is 0 (since the product is 0), another is 2 (since the sum is 2) and it can't happen because of the lemma.

c) $\det(A) = 1$ and $\text{trace}(A) = 2$.

This is a hard case. Characteristic polynomial is $x^2 - 2x + 1 = 0$ so the eigenvalues are equal 1. Let $\mathbf{E}$ be the unit matrix. Then $A - \mathbf{E} = \mathrm{N}$ is a nilpotent matrix. Since the dimension is 2, $\mathrm{N}^2 = 0$. Therefore $A = \mathbf{E} + \mathrm{N}$, and

$$A^{1000} = (\mathbf{E} + \mathrm{N})^{1000} = \mathbf{E} + 1000\mathrm{N}$$

Since all entries of N are less than 11 by absolute values, all entries of 1000N are less than 11000 and all entries of A^{1000} are less than 11001.

d) $\det(A) = -1$ and $\text{trace}(A) = 1$.

Eigenvalues satisfy $x^2 - x - 1 = 0$ so they are $\frac{1 \pm \sqrt{5}}{2}$. The greater eigenvalue is bigger then 1.5, so this case is ruled out by the lemma.

e) $\det(A) = 0$ and $\text{trace}(A) = 1$.

By Cayley-Hamilton, $A^2 - A = 0$, hence $A^2 = A$. By induction. $A^{K+1} = A^K$ and therefore $A^{1000} = A$ and its entries have absolute values no greater than 10.

f) $\det(A) = 1$ and $\text{trace}(A) = 1$.

$A^2 - A + \mathbf{E} = 0$. We multiply that by $A + \mathbf{E}$, and we get $A^3 + \mathbf{E} = 0$.
Therefore $A^3 = -\mathbf{E}$ and $A^{999} = -\mathbf{E}$ and $A^{1000} = -A$. It has the same entries as A, maybe with different signs, but absolute values are still < 10.

g) $\det(A) = -1$ and $\text{trace}(A) = 0$.

$A^2 - \mathbf{E} = 0$. So $A^2 = \mathbf{E}$. Therefore $A^{1000} = \mathbf{E}$.

h) $\det(A) = 0$ and $\text{trace}(A) = 0$.

The matrix is nilpotent. $A^2 = 0 = A^{1000}$.

i) $\det(A) = 1$ and $\text{trace}(A) = 0$.

$A^2 + \mathbf{E} = 0$. So $A^2 = -\mathbf{E}$. Therefore $A^4 = \mathbf{E}$. And $A^{1000} = \mathbf{E}$.