Second stage of Israeli students competition, 2010.

Duration: 4 hours

1. Let $K = \sum_{n \in \mathbb{N}} 2^{-n^2}$ (where $\mathbb{N}$ is a set of positive integers).

Is *K* rational or irrational?

2. ABCD is a tetrahedron (not necessarily regular). Denote *a* = distance between the lines AB and CD, *b* = distance between the lines AC and BD, *c* = distance between the lines AD and BC. Prove that the volume can't be less than $\frac{abc}{3}$.

3. We have a system of *L* lamps and *B* buttons. Each button has one of two states: "on" or "off". Each button is connected to several lamps. A lamp may be connected to more than one button. Pressing a button toggles all connected lamps to the opposite state.

(a) Prove that number of potential states of lamps is a power of 2.

(b) Suppose that for every subset S of lamps there is a button switching state of odd subset of S. Prove that all lamps can be switched off.

4. Compute $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

5. *A* is a 2×2 matrix with integer coefficients. Absolute values of all entries of *A* are less than 10. Absolute values of all entries of A^{1000} are less than 10^9. Prove that they are actually less than 10^6.

Good luck!