Targil 7.

Polynomials and Vieta.

1. Prove that for every polynomial $p(x)$, the polynomial $p(x + p(x))$ is divisible by $p(x)$.

2. Prove that a polynomial $p\big(p(p(x))\big) - x$ is divisible by $p(x) - x$.

3. Six real numbers satisfy:

$$x_1 < x_2 < x_3$$
$$y_1 < y_2 < y_3$$
$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$
$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$
$$x_1 > y_1$$

Prove that $x_3 > y_3$.

4. Three rational nonzero numbers a, b, c are such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$ are integers. Prove that $|a| = |b| = |c|$.

5. The complex roots of polynomial $p(x)$ of degree n are $\alpha_1, \alpha_2, \ldots, \alpha_n$, the complex roots of its derivative are $\beta_1, \beta_2, \ldots, \beta_{n-1}$.
(*a*) Prove that the mass center of $\alpha_1, \alpha_2, \ldots, \alpha_n$ coincides with the mass center of $\beta_1, \beta_2, \ldots, \beta_{n-1}$.
(*b*) Prove that $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are in the convex hull of $\alpha_1, \alpha_2, \ldots, \alpha_n$.