Targil 5.

(Integrals – everybody loves them)

1. Compute $\int_0^{2\pi} \frac{x^2 \sin x}{1+\cos^2 x} dx$.

2. A center of the disc of radius R is on distance d from the axis α, which is parallel to in the plane of the disc. The disc rotates around its center with angular velocity 2 and simultaneously around revolves around axis α with angular velocity 3.
A red point on the boundary of the disc goes along a closed trajectory.
Compute the length of that trajectory.

3.** A smooth function $f:\mathbb{R}^n \to \mathbb{R}$ will be called "nice of order m", if $f(0,0,...,0)=0$ and for any positive integer $k \le m$, for any $i_1, i_2, \ldots, i_k$ we have $\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} ... \frac{\partial}{\partial x_{i_k}} f(0,0,...,0) = 0$.

Consider a function f which is smooth and nice of order m. Prove that there are n continuous functions $f_1, f_2, ..., f_n : \mathbb{R}^n \to \mathbb{R}$, smooth and nice of order $m-1$, such that $f(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} x_k f_k(x_1, x_2, ..., x_n)$.

4.** Let n be a positive integer, and $f:[0,1] \to \mathbb{R}$ a continuous function such that

$$\int_0^1 x^k f(x)\, dx = 1$$

for every $k \in \{0,1,...,n-1\}$. Prove that $\int_0^1 (f(x))^2 dx \ge n^2$.

5. Let $f, g : [a,b] \to (0,\infty)$ be continuous, non-decreasing functions, such that for every $x \in [a,b]$ we have $\int_a^x \sqrt{f(t)}\, dt \le \int_a^x \sqrt{g(t)}\, dt$

and $$\int_a^b \sqrt{f(t)}\, dt = \int_a^b \sqrt{g(t)}\, dt$$

Prove that $$\int_a^b \sqrt{1+f(t)}\, dt \ge \int_a^b \sqrt{1+g(t)}\, dt .$$