Targil 1 - determinants.

1. All entries of a 10×10 matrix A belong to the interval [-1,1].
Can it happen that $\det A > 5770$?

Solution. We shall use a following lemma:

Lemma. If $N = 2^k$, we can construct a square $N \times N$ matrix which consists of numbers 1 and -1 and its columns are mutually orthogonal.

Historical remark. A matrix of that kind, with mutually orthogonal columns and made of ones and minus ones, is called Hadamard matrix after a famous French (Jewish) mathematician Hadamard (pronounced [adamAr]).
It is easy that the size of Hadamard matrix, if it is bigger than 2, should be divisible by 4; Hadamard conjecture is that for any N divisible by 4 there exists an Hadamard matrix of that size. The first nontrivial examples, of size 12 and 20 were constructed by Hadamard in 1893; Wikipedia lists several other constructions for different sizes, the last of those is the construction in 2004 by Hadi Kharaghani and Behruz Tayfeh-Rezaie of size 428; existence for size 668 is still an open.
http://en.wikipedia.org/wiki/Hadamard_matrix

Suppose we have proved the lemma; let's solve our problem in this case.
Determinant of Hadamard matrix is $\pm N^{N/2}$, and we may choose the sign: if we want to change it, we multiply one of the rows by -1. Indeed, determinant is an oriented volume of a parallelepiped (מקבילון) spanned by the vectors of the columns; our parallelepiped is a cube, sides are of length $\sqrt{N}$. So the volume, up two a sign is a product of sides.
So the determinant of Hadamard 2×2 matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is 2, and the determinant of Hadamard 2×2 matrix is $8^4 = 2^{12} > 4000$ and if we put those two matrixes as blocks along the diagonal we get a 10×10 matrix with det > 8000 > 5770.
Now it remains to prove the lemma.

Proof of lemma. In our case, orthogonality means that each two columns match in precisely half of the places (and mismatch in the other half). We shall construct a matrix of zeroes and ones with the same combinatorial properties; to turn it into Hadamard matrix, it will be enough to replace zeroes by -1.
Since $N = 2^k$, the rows, as well as the columns, will be numbered by a different vectors of k-dimensional space over the field of two elements $\{0, 1\}$.
The number in the column that corresponds to vector v and row that corresponds to vector u will be the scalar product of v and u.
Consider two different columns corresponding to binary vectors v and w. The numbers in these columns in row corresponding to vector u match, iff (if and only if) u is orthogonal to $v - w$. Orthogonality to $v - w$ is a condition, that specifies a hyperplane in the linear space over the field of two elements. Every hyperplane in that space contains precisely ½ of its vectors; adding any vector outside the hyperplane is a bijection (העתקה חח"ע ועל) between the hyperplane and its complement. QED.

Remark. Another way to do this construction is by induction.
A shorter, but also a less elementary way, for those who know tensor product, is to take n'th tensor power of the 2×2 Hadamard matrix,

2. Matrix B has zeroes on the main diagonal and ones at all other places.
a. Prove that B is non-degenerate.
b. Compute det B.

Solution. a. Sum of all columns divided by something is a vector of ones.
Difference between that and some column is an element of the standard basis.
So, any element of a standard basis can be spanned by the columns of our matrix; hence the whole space is spanned by the columns of our matrix.
b. If I add a unit matrix to B, I get a matrix of rank 1, therefore –1 is an eigenvalue of multiplicity $n - 1$ at least. If I subtract n – 1 times unit matrix, I get the following matrix: 1 – n on the diagonal, and 1 elsewhere. Sum of all columns is a zero vector, hence the matrix is degenerate. Hence $n - 1$ is also an eigenvalue.
We have found n eigenvalues ($n - 1$ times –1, and once $n - 1$); that's the complete list of eigenvalues; the determinant = product of all eigenvalues = $(-1)^{n-1}(n - 1)$.

3. a. Compute $u = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 2^3 & 3^3 & 4^3 & 5^3 \end{pmatrix}$

b. Prove that $v = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^4 & 3^4 & 4^4 & 5^4 \\ 2^7 & 3^7 & 4^7 & 5^7 \end{pmatrix}$ is divisible by u.

3. a. We shall compute the more general determinant:

$$u = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

(sometimes a more general question is easier than a special case).
The determinant is 0 when $b = a$, so it is divisible by $b - a$.
For similar reasons, it is also divisibly by $c - a$, by $d - a$, … .
So, we have a polynomial in a, b, c, d, which is divisible by

$$(b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

The last determinant, as well as u, is a polynomial of degree 6 (in each permutation, we have something of degree 1 times something of degree 2 times something of degree 3). Hence

$$u = K(b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

where K is a constant. The only thing that we have to do yet is to guess a constant. When we compute u by a sum of permutation, we have only one way to get highest (third) power of d, and second power of c, and first power of b; by the diagonal permutation. When we open brackets in the product, the only way to get this (highest powers of the latest variables) is to take all plusses out of brackets.
In both cases, we get 1 as the coefficient, hence $K = 1$.

$$u = (b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

In our case, we get $u = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 = 12$.

Remark. The same thing that was done for the 4×4 matrix can be done for $N \times N$ matrix in the same way. If the k'th row from matrix k contains powers of x_i, from 0 power = 1 to the $N-1^{st}$ power, the determinant is $\prod_{i<j}\left(x_j - x_i\right)$.

The proof is precisely the same: notice that it is divisible by $x_j - x_i$, compare degrees (which are equal), and find the coefficients.
This nice determinant is called Vandermonde.

b. By similar reasons, $v = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^4 & b^4 & c^4 & d^4 \\ a^7 & b^7 & c^7 & d^7 \end{pmatrix}$ is divisible by (b – a), (c – a), etc.

hence it is divisible by the product $u = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$.
The ratio is a symmetric polynomial in a, b, c, d.

Well, actually in the above proof we have used here some non-obvious facts, which I told this yom revii in the classroom (when we talked of resultants).
I shall not prove them now in a detailed way, but I shall split it into a sequence of very easy (but not completely obvious) lemmas:

Lemma 1. Consider a polynomial of one variable over an infinite field. Suppose its value at every point is zero. Then it's the zero polynomial (all coefficients are 0).
Lemma 2. Consider a polynomial of several variable over an infinite field. Suppose its value at every point is zero. Then it's the zero polynomial.
Lemma 3. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ has zero values at all points of a hyperplane $\{ x_1 = 0 \}$. Then the polynomial is divisible by x_1 (it can be written as x_1 times another polynomial).
Lemma 4. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ has zero values at all points of a hyperplane $\{ l(x_1, x_2, \ldots, x_n) = 0 \}$, where l is a linear function. Then the polynomial is divisible l.
Lemma 5. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ with integer coefficients has zero values at all points of a hyperplane $\{ l(x_1, x_2, \ldots, x_n) = 0 \}$, where l is a linear function with integer coefficients, and coefficient of x_k is 1. Then the polynomial is l as a polynomial with integer coefficients.

4*. The inverse and the determinant of the following matrix: $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$.

Solution. Once again, consider a more general problem N×N matrix, in column i row j we have the number $\frac{1}{a_i + b_j}$. Our case is $N = 4$, $a_i = i$, $b_j = j - 1$.

Unfortunately, the entries of the matrix and its determinant art not polynomials. To make everything polynomial, we should multiply each column by all denominators there, thus making a matrix terrifying, but polynomial. The determinant will be multiplied by the common denominator $\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)$ which is a polynomial of degree N^2. The determinant originally was a rational function of degree $-N$, so now it is a polynomial of degree $N\cdot(N-1)$.

The determinant is 0 when $a_i = a_j$, and when $b_i = b_j$, hence it is divisible by $\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)$. The degrees coincide, so that is an answer up to a constant factor. So the original determinant is $\frac{\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)}{\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)}$ up to coefficient.

To compute the coefficient, let us take a special case: $a_i = i, b_i = \varepsilon - i$, where ε is a very small positive number. All entries of the matrix outside the main diagonal are decent bounded numbers, while on main diagonal we have $\frac{1}{\varepsilon}$. The determinant is a very big number: $\left(\frac{1}{\varepsilon}\right)^N + O\left(\left(\frac{1}{\varepsilon}\right)^{N-2}\right)$. Now estimate the second expression:

$$\frac{\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)}{\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)} = \frac{\prod_{i<j}(j-i)(i-j)}{\prod_{i=1}^{N}\prod_{j=1}^{N}(\varepsilon + j - i)} \approx \frac{\prod_{i\neq j}(j-i)}{\varepsilon^N \times \prod_{i\neq j}(j-i)} = \frac{1}{\varepsilon^N}$$

So, both expressions are of asymptotically $\approx \frac{1}{\varepsilon^N}$ therefore the coefficient between them is 1. Hence the determinant is $\frac{\prod_{i<j}(a_j - a_i)(b_j - b_i)}{\prod_{i=1}^{N}\prod_{j=1}^{N}(a_i + b_j)}$.

The matrix made from $\frac{1}{a_i + b_j}$ is called Cauchy matrix (after a famous French mathematician). The special case of $\frac{1}{i + j - 1}$ is called Hilbert matrix (after a famous German mathematician). In our special case of 4×4 matrix of size 4 we get:

$$\det = \frac{(1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3)^2}{1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^3 \cdot 6^2 \cdot 7}$$

Now about inverse matrix. There is a formula for that: the element row *k*, column *l* of inverse matrix is

$$r_{k,l} = (-1)^{k+l} \frac{\det A_{l,k}}{\det A}$$

where $A_{l,k}$ (aka minor) is a matrix without row *l*, and column *k*.

But minors of a Cauchy matrix are again Cauchy matrix:

$$r_{k,l} = (-1)^{k+l} \frac{\det A_{l,k}}{\det A} = (-1)^{k+l} \frac{\prod_{\substack{i<j \\ i,j \neq l}}(a_j - a_i) \prod_{\substack{i<j \\ i,j \neq k}}(b_j - b_i)}{\prod_{\substack{i=1 \\ i \neq l}}^{N}\prod_{\substack{j=1 \\ j \neq k}}^{N}(a_i + b_j)} \cdot \frac{\prod_{i=1}^{N}\prod_{j=1}^{N}(a_i + b_j)}{\prod_{i<j}(a_j - a_i)(b_j - b_i)} =$$

$$= \frac{\prod_{i \neq l}(a_i + b_k)\prod_{j=1}^{N}(a_l + b_j)}{\prod_{i \neq l}(a_l - a_i)\prod_{j \neq k}(b_k - b_j)}$$

Specifically for Hilbert matrix we get $r_{k,l} = \frac{\prod_{i \neq l}(i + k)\prod_{j=1}^{N}(l + j)}{\prod_{i \neq l}(l - i)\prod_{j \neq k}(k - j)}$.

5.** Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients.

Show that the determinant is a perfect square.

Remark. $\det A = \det A^T = (-1)^n \det A$, so it is nonzero (and non-obvious) only for even dimension.

First solution. Determinant is integer, so it is enough to prove the it is a square of rational number, then we shall know it is a square of integer. If we apply a certain permutation on rows and the same permutation on columns, matrix will remain anti-symmetric and will keep the same determinant.

So we may assume that unless the matrix consists of zeroes only, then cells near the left-top corner (1,2) and (2,1) are non-zero: one is a, another is $-a$. Then by adding linear combinations of first and second rows to all other rows, we can eliminate all numbers in the first and second columns after the second row. These Gauss method operations are equivalent to multiplying the matrix from the left by an invertible matrix.

If A is anti-symmetric, then it is easy to see that BAB^T is also anti-symmetric. Let B be the matrix that is doing Gauss method operation to eliminate the first two columns under the top-left 2×2 block. Then B^T does the same operations on the columns. Obviously, both B and B^T are rational, so determinant is multiplied by a square of rational number. That number is nonzero, since B is invertible.

But now we get a block matrix, that consists of 2 anti-symmetric blocks, so the statement follows by induction over dimensions.

Second proof. It is known, that over anti-symmetric multi-linear forms the wedge product is defined, that makes a $k+m$-form out of k-form and m-form.

$$(\kappa \wedge \mu)(v_1, v_2, ..., v_m) = \frac{1}{k!m!} \sum_{\sigma \in \mathbb{S}_{k+m}} \operatorname{sgn}(\sigma) \kappa\left(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}\right) \cdot \mu\left(v_{\sigma(m+1)}, v_{\sigma(m+2)}, ..., v_{\sigma(m+k)}\right)$$

(here we divide by $k!m!$ to kill ambiguity – no need to sum equivalent summands several time, so this formula is actually integer).

This product is super-commutative and associative.

Any anti-symmetric 2-form can be represented in a general form as $\sum_{i<j} a_{ij} x_i \wedge x_j$,

where x_i are basic linear functionals corresponding to "taking i'th coordinate", or, when suitable basis is chosen, in a canonic form:
$\omega = k_1 x_1 \wedge x_2 + k_2 x_3 \wedge x_4 + k_3 x_5 \wedge x_6 + ... + k_n x_{2n-1} \wedge x_{2n}$.

Actually, that was what we have proven in the first solution.

But since the definition of the wedge product doesn't use coordinates, as well as some definitions of determinant, if we prove certain equality between those in the canonical basis, we shall know it for any basis.

Consider the product $\frac{\omega \wedge \omega \wedge ... \wedge \omega}{n!}$, where ω is multiplied by itself n times.

When we open brackets, all products with similar factors cancel out. So we get $n!$ equivalent products, so after dividing by $n!$ we get an expression which is integer and not fractional in the coefficients, and that is $(k_1 k_2 k_3 ... k_n) x_1 \wedge x_2 \wedge x_3 \wedge ... \wedge x_{2n}$, product of all coefficients time standard volume form.

The determinant of the anti-symmetric matrix is $k_1^2 k_2^2 k_3^2 \cdot ... \cdot k_n^2$. It is the square of the coefficient before the volume form of $\frac{\omega \wedge \omega \wedge ... \wedge \omega}{n!}$. So it will be not necessarily in the canonical basis.

Example. Consider $n = 4$. Matrix $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$ is represented by a form $\omega = a_{12} x_1 \wedge x_2 + a_{13} x_1 \wedge x_3 + a_{14} x_1 \wedge x_4 + a_{24} x_2 \wedge x_4 + a_{23} x_2 \wedge x_3 + a_{34} x_3 \wedge x_4$.

Then $\frac{\omega \wedge \omega}{2} = (a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}) x_1 \wedge x_2 \wedge x_3 \wedge x_4$.

(When computing this things, just multiply each couple of terms once and don't divide by 2).

So $\det A = \left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)^2$.

Outline of third solution (Ofir Gorodetzky)

We know (either by guessing or from previous solution) the formula for the expression whose square is the determinant: it is a sum over all ways to decompose the set of all indices into pairs, of product of cells corresponding to that pairs (one index is of row, another of column), signs are chosen by the sign of a permutation which is formed when we write down all those pairs in a row, pair after pair.

So, we can prove combinatorially, that the square of that expression is the determinant. The determinant is a sum of all products over all permutations (or maximal rook arrangements). Some of those permutations contain odd cycles, others only even cycles. We can show that any permutation containing at least one odd cycle will cancel out with another permutation because the matrix is anti-symmetric (by transposing only that specific cycle).

So, we remain with permutations having even cycles only. Sides of even circle might be colored into black and white. That splits the permutation into two perfect matchings. Each of those perfect matchings can be considered as a summand in the polynomial we described, so the determinant is what we get after multiplying that expression by itself (since each time we unite 2 pair decompositions, we get a permutation with even cycles). Working out the signs is left as an exercise ☺.