Targil 11.

This targil is about identities. It is much harder to find a nice non-trivial identity than a nice non-trivial inequality; still there are some.

1. Prove that for 2×2 matrixes $\det(A)=\frac{(\mathrm{tr}(A))^2-\mathrm{tr}(A^2)}{2}$ and, more generally, for $n\times n$ matrixes $\det(A)$ can be expressed as a polynomial in $\mathrm{tr}(A),\mathrm{tr}(A^2),...,\mathrm{tr}(A^n)$.

2.** Prove that for arbitrary $2n$ matrixes $A_1, A_2, \ldots, A_{2n}$ of size $n\times n$,

$$\sum_{\sigma\in S_{2n}}(-1)^{\mathrm{sgn}(\sigma)}A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)}\cdot\ldots\cdot A_{\sigma(2n)}=0.$$

(Here S_{2n} is a group of permutations of 2n numbers).

3. Prove: $\sum_{i=1}^{n}\frac{x_i^m}{\prod_{j\neq i}(x_i-x_j)}=\begin{cases}0, \text{ for } m=0,1,...,n-2\\ 1, \text{ for } m=n-1\end{cases}$.

4.* We are given an orthogonal matrix $n\times n$; let d_1 be the determinant of its $k\times k$ upper left corner, and d_2 be the determinant of its of its $(n-k)\times(n-k)$ right bottom corner. Prove that $|d_1|=|d_2|$.

5.* Prove: $(1-x)(1-x^2)(1-x^3)\cdot\ldots=1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{3k^2-k}{2}}+x^{\frac{3k^2+k}{2}}\right)$

6.** Prove $p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{k-1}(r+kq)^{n-k}=(p+r)^n$.