Targil 10.

This targil is about probability.

1. Prove that for any $0 < p < 1$, and integers $m, n > 1$,

$$(1 - p^n)^m + (1 - (1 - p)^m)^n > 1.$$

Solution. Consider a $m \times n$ table – it has n rows, m columns.

In every cell, we write 1 with probability p, and 0 with probability $1 - p$.
So, a probability that a given column row doesn't consists of ones is $1 - p^n$ (or, in other words, given column has zero). The probability that every column has zero is $(1 - p^n)^m$. Similarly, the probability that every row has one is $(1 - (1 - p)^m)^n$.
It is not possible that neither of this two events take place: it'd mean we have both a column of zeroes, and a row of ones, and then in the intersection of these two would be a contradiction. This gives a non-strict inequality.
It is easy to find a table where both events take place simultaneously; hence the inequality is strict.

2. The floor is tiled by $l \times l$ squares (so it looks like a lattice). You throw a needle of length n on the floor. What is the chance that when it falls, it won't cut the lines, but will be entirely within one of the squares?

Solution. Let us start with one-dimensional case: a needle of length n is thrown on the real line, which is tiled by intervals of length l. The probability of hitting a joint is 1 if $n \geq l$ and is n / l if $n < l$.
Now we drop a needle on the two-dimensional floor. It is rotated by angle α, which is random (and uniformly distributed). Given α, we can split the needle into two projection: it has length $|n \cos \alpha|$ in x direction, and $|n \sin \alpha|$ in y direction.

The probability of hitting vertical lines is $p_h = \max\left(\left|\frac{n\cos\alpha}{l}\right|, 1\right)$, and of hitting horizontal lines is $p_v = \max\left(\left|\frac{n\sin\alpha}{l}\right|, 1\right)$.

There are 3 cases of different nature.

a. $n \geq l\sqrt{2}$, then the probability of hitting the grid is 1, and nothing to compute.

b. $l\sqrt{2} > n > l$, then at some angles probability is 1, at other angles probability is analytic expression.

c. $l \geq n$, then for each angle probability can be computed as a fraction (without maximum).

We shall start with case c.

The needle doesn't hit the grid if it doesn't hit neither vertical nor horizontal lines, which is $\left(1-\left|\frac{n}{l}\cos\alpha\right|\right)\left(1-\left|\frac{n}{l}\sin\alpha\right|\right)$. This probability must be averaged along the circle:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(1-\tfrac{n}{l}\left|\cos\alpha\right|\right)\left(1-\tfrac{n}{l}\left|\sin\alpha\right|\right)d\alpha = \frac{4}{2\pi}\int_0^{\pi/2}\left(1-\tfrac{n}{l}\cos\alpha\right)\left(1-\tfrac{n}{l}\sin\alpha\right)d\alpha =$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left(1-\tfrac{n}{l}\cdot 2\sin\alpha+\tfrac{n^2}{2l^2}\sin(2\alpha)\right)d\alpha = 1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_0^{\pi/2} = .$$

$$= 1+\frac{2}{\pi}\left(\tfrac{n^2}{2l^2}-\left(\tfrac{n}{l}\cdot 2-\tfrac{n^2}{2l^2}\right)\right) = 1+\frac{2}{\pi}\left(\tfrac{n^2}{l^2}-\tfrac{n}{l}\cdot 2\right)$$

Hence the probability of hitting a line is $\frac{2}{\pi}\cdot\frac{n}{l}\left(2-\frac{n}{l}\right)$.

Now for the case b. There's a range of angles, which are close to the integer multiples of $\frac{\pi}{2}$, which give 1 as probability for hitting.

These probability one situations should be of length no more than ϕ from either vertical or horizontal direction, where $\phi = \arccos(l/n)$. So, if slope is between 0 and $\frac{\pi}{2}$ (it is something we can assume because of symmetry), then the probability of having angle that implies hitting is $\frac{2\phi}{\pi/2} = \frac{2}{\pi}\arccos(l/n)$.

To compute the rest of the probability in this case, we have to compute integral similar to the previous case, from ϕ to $\frac{\pi}{2}-\phi$.

$$\frac{2}{\pi}\int_{\phi}^{\frac{\pi}{2}-\phi}\left(1-\tfrac{n}{l}\cos\alpha\right)\left(1-\tfrac{n}{l}\sin\alpha\right)d\alpha=\frac{2}{\pi}\int_{\phi}^{\frac{\pi}{2}-\phi}\left(1-\tfrac{n}{l}\cdot 2\sin\alpha+\tfrac{n^2}{2l^2}\sin(2\alpha)\right)d\alpha$$

$$=\frac{2}{\pi}\left(\frac{\pi}{2}-2\alpha\right)+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_{\phi}^{\frac{\pi}{2}-\phi}$$

Hence the complete probability of hitting is

$$p=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_{\phi}^{\frac{\pi}{2}-\phi}=$$

$$=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{2l^2}\cos(2\phi)-2\cos\phi+\tfrac{n^2}{2l^2}\cos(2\phi)\right)=$$

$$=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{l^2}\cos(2\phi)-\tfrac{n}{l}2\cos\phi\right)$$

Now recall that $\cos\phi=\dfrac{l}{n}$, $\sin\phi=\sqrt{1-\dfrac{l^2}{n^2}}$, $\cos 2\phi=2\dfrac{l^2}{n^2}-1$.

$$p=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{l^2}\cos(2\phi)-\tfrac{n}{l}2\cos\phi\right)=$$

$$=1+\frac{2}{\pi}\left(\frac{n}{l}\cdot 2\sqrt{1-\frac{l^2}{n^2}}+\frac{n^2}{l^2}\left(2\frac{l^2}{n^2}-1\right)-2\right)=$$

$$=1+\frac{2}{\pi}\left(\frac{2\sqrt{n^2-l^2}}{l}-\frac{n^2}{l^2}\right)$$

In such messy computations, it is recommended ***to verify extreme cases***.

For instance, when $\dfrac{n}{l}=\sqrt{2}$, the result should be 1; for $n = l$ the result should be the same as in the case c.

Remark. Anyway, if the length of the needle < the diagonal of the tile, we get an experimental method of measuring π.

3. For any natural numbers *k*, *m*, *n* compute the integral

$$\int_0^1\left(\int_0^{1-y} y^k x^m\left(1-x-y\right)^n dx\right)dy.$$

Remark. Actually, this is an analogue of beta-function (of dimension +1).

Beta function is the integral $\int_0^{1-y} x^m (1-x)^n \, dx$ which is classical Euler's example of for integration by parts (and the answer is almost the reciprocal of binomial coefficient, as in our problem the answer is almost the reciprocal of trinomial coefficient), and of course, this exercise follows from it. However, both ideas below can be applied to the beta function.

First solution. We can conclude this integral from another famous Euler's integral, $\int_0^{\infty} x^n e^{-x} dx = n!$ (a. k. a. gamma function, up to shift by one). The proof of that is a simple exercise in induction and integration by parts.

No consider three-dimensional integral:

$$k! \cdot n! \cdot m! = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^k e^{-x} \cdot y^m e^{-y} \cdot z^n e^{-z} dxdydz = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^k y^m z^n e^{-(x+y+z)} dxdydz$$

Now switch to another variables $t = x + y + z$, $u = x/t$, $v = y/t$,

$$u = xt \text{ , } v = yt \text{ , } z = (1 - x - y)t.$$

The Jacobi matrix is $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} t & 0 & x \\ 0 & t & y \\ -t & -t & 1-x-y \end{pmatrix}$.

Adding first two rows to the last brings us to the matrix $\begin{pmatrix} t & 0 & x \\ 0 & t & y \\ 0 & 0 & 1 \end{pmatrix}$, so the determinant is t^2. Hence the integral that we computed was

$$k! \cdot n! \cdot m! = \int_0^{\infty} \iint_{\substack{u,v \ge 0 \\ u+v<1}} (ut)^k (vt)^m \left(t(1-u-v)\right)^n e^{-t} t^2 dudvdt =$$

$$= \iint_{\substack{u,v \ge 0 \\ u+v<1}} u^k v^m (1-u-v)^n \, dudv \cdot \int_0^{\infty} t^{k+m+n+2} du = X \cdot (k+m+n+2)!$$

Where X denotes the integral we wanted to know from the beginning.

Therefore, $X = \dfrac{k!\cdot n!\cdot m!}{(k+m+n+2)!}$.

Remark. In the definition of both beta function and gamma function the exponents are always $n - 1$ and not n. I think the aesthetic reason for that definition was that people prefer to get the formula $\mathrm{B}(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ and not m + n + 1 below.

There are also some ideological reasons for this (they say it is Mellin transform, which is version of Fourier for multiplicative group of positive numbers, but these reasons appeared much later. The next solution is even nicer than the previous, but works only for integer number (while the previous is easily generalizable for complex numbers).

Second solution. We put on the [0,1] interval, randomly (with uniform probability measure) and independently, $N + 2$ dots: two red and N blue dots.

The first red dot is located at x, the second at $1 - y$, on distance y from the right end; the distance between them is $1 - y - x$. The distribution of x and y is uniform on the triangle $\{x, y \geq 0\ ,\ x + y \leq 1\}$, so the probability measure is $2dxdy$ supported on that triangle (the two red dots are distributed uniformly over the square, and then we impose the condition that first is before the second, we fold the square).

Now, when the red dots divided the segment into 3 sub-segments, So, let us compute the probability that m first blue points are in the left sub-segment, n next blue points in the middle sub-segment, and k last blue points are in the right sub-segment (here $k + m + n = N$).

If the red points are already placed, the probability is $y^k x^m (1-x-y)^n$.

So if red points are not yet placed, we get $2\int_0^1 \left(\int_0^{1-y} y^k x^m (1-x-y)^n \, dx \right) dy$.

Now let us compute the same probability in another way.

Each point-placing experiment gives us a random order of points, and each order has the same probability: $\dfrac{1}{(N+2)!}$. Out of this orders some are considered good we have m points to place on m first places, other n points for n places, k more

points for k places, and 2 more red points for 2 splitting places, so 2!k!m!n! orders are good, and the probability is $\frac{k!m!n!2}{(k+m+n+2)!}$. Therefore:

$$2\int_0^1\left(\int_0^{1-y} y^k x^m (1-x-y)^n\,dx\right)dy = \frac{k!m!n!2}{(k+m+n+2)!}$$

$$\int_0^1\left(\int_0^{1-y} y^k x^m (1-x-y)^n\,dx\right)dy = \frac{k!m!n!}{(k+m+n+2)!}$$

4.* **a.** Four aliens land on the surface of a spherical planet, randomly and independently (a chance for every alien to land inside a country is proportional to the area of this country). What is the probability that they all landed on the same hemisphere? In other words, what's the probability that there exists a plane through the planets center, such that all landing points are on the same side of that plane?
b. Ten aliens land on the planet (independently, proportional to surface area). What is the probability that they are on the same hemisphere?

Solution. For simplicity, we assume that the radius is 1. Also, we shall talk about open hemispheres – the chance of landing on the edge of a hemisphere is 0.
We can choose n random points of the sphere with the following procedure: first choose n random diameters of the sphere, then we shall choose one end of each diameter. If we would choose a given hemisphere, the chance that all points are in it would be $1/2^n$, for almost any choice of diameters.
There is a natural one-to-one correspondence between hemispheres and points of the sphere – in spherical geometry, hemisphere is a disc of radius $\frac{\pi}{2}$, and hence it is specified by its center. Another way to say the same thing – hemisphere is a set of vectors on sphere, which have positive scalar product with a given vector; and that vector corresponds to the hemisphere.
So, assume we have n diameters. To each diameter we draw a perpendicular bisector. This gives us n planes; each of this planes cuts the sphere along a big circle. The big circles cut the sphere into K parts, we shall compute K later. If we choose a center of hemisphere in one of these parts, probability that all will land in

it will be $1/2^n$. Events of landing on the given hemisphere are the same if centers are in the same part, and disjoint if they are in different parts. Therefore the probability of being in the same hemisphere is $K/2^n$.
Now it is enough to compute K. First big circle cuts the sphere in two. When we have already m circles, any new circle cuts other circles in $2m$ different points (with probability 1), so we add $2m$ new arcs, each splitting existing region into two new regions; therefore number of regions is increased by 2.
Therefore, if there are 4 points, number of regions is 2 + 2 + 4 + 6 = 14, and the probability is $14 / 2^4 = 7/8$.

For 10 points, number of regions is $2+2+4+\ldots+18=2+\frac{0+18}{2}\cdot 10=2+90=92$,

and the probability is $92 / 2^{10} = 23 / 256$.

5.** A unit cube is orthogonally projected on a random plane (the normal vector to the plane is distributed uniformly over the unit sphere).
What is the expectation of projection area?

Answer. 3/2.
Solution. Because the cube is a convex polytope, area of its projection to any plane is half sum of areas of projections of all faces. So, the expectation of the area is 3 times the expectation of projection area of a unit square.
The area of projection of a planar shape to another plane is the original area before the projection times the absolute value of the cosine. In our case, the original area is 1, so we have only the cosine.
Since the area of spherical hat is proportional to its height, we get that expectation of the area of a unit square projection is ½.
Therefore the expectation of area of cube projection is $3/2$.