First stage of Israeli students competition, 2009.

1. Let C be a convex polygon and P a point inside it. Let N be number of vertices, such that an interval connecting P to the vertex divides the angle of C into two acute angles. Denote n number of sides of C, such that the foot of perpendicular from P to that side is strictly inside that side.
Proof that $N = n$.

First solution. Consider function of a point $f(X) = |PX|$, defined on the perimeter of the polygon.
It is easy to see that n is number of minima of that function, and N is the number of its maxima. When you go clockwise on the polygons perimeter, there is one maximum after each minimum before the next minimum, and one minimum after each maximum before the next maximum, so number of maxima = number of minima.

Second solutions. Let M be the number of vertices of the polygon. Then the intervals connecting P to the vertices split the angles of the polygon into 2M angles, and assume K of those angles are not acute.
Then N = M – K, because only one of two angles at each vertex can be not acute.
The foot of perpendicular from P to a certain side is inside, if both angles which are adjacent to this side are acute. At least one of them is always acute.
Therefore n = M – K also.

2. Let A be a 2×2 matrix with real coefficients. One of its coefficients is 200.
Can it happen that all the coefficients of matrices A^{-1}, A^2, A^3, A^4, …, A^{100} belong to the interval (-10, 10) ?

Answer. No.

Solution. $A = A^{-1}A^2$, hence each coefficient A is a sum of two products of a coefficient of A^{-1} and A^2, which both have absolute value less than 10.

So, each of those two products has an absolute value less than 100 and the sum has absolute value less than 200.

3. The sequence $\{x_i\}$ is defined by the initial value $x_0 \subset [0,1]$ and recursive formula $x_{n+1} = \dfrac{1-\sqrt{1-x_n}}{2}$.

Find $\lim\limits_{n\to\infty}\left(x_n \cdot 4^n\right)$.

Answer. $\left(\arcsin\sqrt{x_0}\right)^2$.

Solution. Denote $x_0 = \sin^2\alpha$, where $0 \le \alpha \le \dfrac{\pi}{2}$. Then

$$x_1 = \frac{1-\sqrt{1-\sin^2\alpha}}{2} = \frac{1-\cos\alpha}{2} = \sin^2\frac{\alpha}{2}$$

By repeating this idea several times, we get $x_n = \sin^2\dfrac{\alpha}{2^n}$.

When n tends to infinity, $\dfrac{4^n x_n}{\alpha^2} = \left(\left(\sin\dfrac{\alpha}{2^n}\right)\Big/\left(\dfrac{\alpha}{2^n}\right)\right)^2 \to 1$, therefore

$$\lim_{n\to\infty}\left(x_n \cdot 4^n\right) = \alpha^2 = \left(\arcsin\sqrt{x_0}\right)^2.$$

4. Two players play a game on the infinite chess-board. First player plays with 3 white pieces called sheep, and the second player plays with 3 black pieces, called wolves. They move in turn. In his move each player can move only one piece to an adjacent cell (having a common side with its previous cell). Sheep can be moved only horizontally. If a wolf and a sheep happen to be in the same cell, the wolf eats the sheep. Is it always possible for wolfs to catch at least one sheep?

First solution. The sheep remain always on the same horizontal lines, so it is possible to move the wolves so that each sheep would have a wolf on the same horizontal line. Then each sheep would have a wolf strictly to the left or strictly to the right with respect to it. At least two sheep will have a wolf at the same side. WLOG (without loss of generality we can assume that) there are two sheep that have wolves on the left.

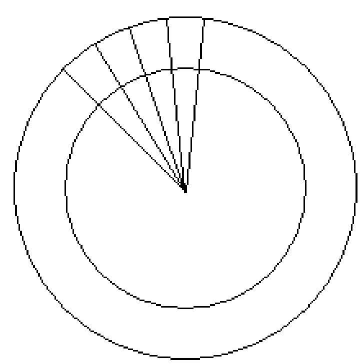
Then, the leftmost wolf of those two will do N moves to the right, where N is a very large number. The sheep in the same row will have to run away, that is, most of the moves (all except M moves, where M is the distance between the wolf and the sheep in that row) will be by this sheep and right. So, if N is much bigger than N the wolf will be far to the right with respect to the other sheep.

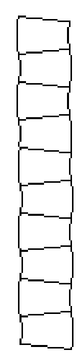
Then the same wolf will make several vertical moves, to arrive to the row of another sheep having wolf on the left. Since N is large, and distance between the rows is bounded, the wolf will turn out to be on the right of that sheep. So, that sheep will have a wolf on each side. Now, if those wolves move to meet each other, they shall eat that sheep before meeting.

Second solution. Let N be a number greater than distances between the pieces in both X and Y directions. The leftmost wolf will make 30N move to the left and then the rightmost wolf will make 30N moves to the right. During that time, the sheep will make 60N moves, so there will be a sheep that have made no more than 20N moves. So, leftmost wolf is at least 10N to the left and rightmost wolf is at least 20N to the right. It will take no more than 2N moves for those 2 wolves to arrive to the horizontal line of that sheep, and then the sheep will be between them.

Now, if those wolves move to meet each other, they shall eat that sheep before meeting. And here only 2 wolves were used!

5. When in three-dimensional space the center of the ball of radius *r* goes along a circle of radius *R* (here $R>r>0$), the ball covers a three-dimensional body called torus. Compute the surface area of that torus as a function in *r* and *R*.





Answer. $4\pi^2Rr$.
First solution. Draw many planes containing the axis of rotation, at the same angles. Those planes cut the torus into many thin bands, almost cylindrical but skew (the inner side will be shorter than the outer side).
Now take those bands apart and glue them in vertically in alternating directions. We get a tube, which is almost cylindrical. The tangent to the circle in close points will have similar directions; that is why the tangent planes to this almost cylindrical tube will be almost vertical, if bands are thin enough.

Take a limit when the angles between consequent planes go to 0. The area is still the same, the length around the cylinder is $2\pi r$, and the height is $2\pi R$, so the surface of cylindrical tube is $4\pi^2 Rr$ and the height is the same.

Second solution. We shall prove a more general statement. Consider a planar curve of length l in one side of the line in the same plane. The distance between the line and the mass center of the curve is d. Then the volume of rotation surface, created by rotating that curve around that line is $2\pi ld$.
Our problem is obviously a special case of that general statement.
Let us assume that the rotation axis is y axis, and the plane of the curve is xy, and the curve is on the right of the axis (positive x).
To prove the theorem, divide the curve into infinitesimal subintervals (or very small subintervals which are almost linear and then take limit of sum, which is integral). The interval of length dl on distance x from the axis is forming a strip of area $2\pi x$ dl, since its length is $2\pi x$ and its width is constantly dl.
So the total area is integral over the curve of $2\pi x$ dl.
But the integral of x dl is precisely the length of curve times x-coordinate of the mass center, by the definition of the mass center.
Therefore the total area is l times $2\pi d$, since d is distance between the mass center of the curve and the axis. QED.

Remark. One can prove similar statements about volume of rotation body – it is equal length of the mass center orbit times area of the original planar form.