Third stage of Israeli students competition, 2009.

1. Denote A be number of ways to paint the cells of the 8×8 chessboard in 3 colors, so that no two adjacent cells are of the same color (by adjacent cells we mean cells having common side). Denote X the number of ways to write integer numbers in the cells of the chessboard, so that the number in the bottom left corner is 0, and the difference between numbers in any two adjacent cell is 1 (here by difference of x and y we mean $|x - y|$).
Express X via A.

Answer. $X = A / 3$.
Solution. Assume we have a table of numbers, satisfying the condition. If we paint the cells having numbers of type $3k$ red, cells having numbers of type $3k+1$ green, and cells having numbers $3k+2$ blue, we get a coloring of the board in 3 colors, and the left-bottom cell a1 is red.
So, we get a coloring of the board satisfying the condition, with the specified color at a1, and that is only 1/3 of all possible colorings (it is easy to see that all colors for a specific cell have equal probabilities, since we can rotate colors, by replacing red → green → blue → red).
So, each one of permitted X tables of numbers can be turned into one of A/3 coloring. It remains to prove that this correspondence is 1-1. To show it, we should explain why given a coloring of the board we can reconstruct – and uniquely – the table of numbers.
Firstly, if we know the coloring and we know the number at a certain cell, we can reconstruct the number at an adjacent cell. That is because we have only 2 options ($x + 1$ or $x - 1$, where x is the number in the first cell), and these two options have different remainders mod 3, so the coloring allows us to distinguish those two options. Notice, that each of two remainders different from x are attainable, and they correspond to both colors different from the color of x.
We start with the cell a1 and write 0 in that cell, since this is given. Then, as we described before, we can reconstruct numbers of b1, c1, … , h1 and also numbers a2, a3, … , a8.
From now on we shall reconstruct b2, b3, …, b7, c2, c3, …, c7 and so on. As we saw before, reconstruction given one neighbor and color exists and unique, so the question is whether reconstruction given 2 neighbors (down and left) and color exists, because if it is, it is also unique. Suppose WLOG that we try to reconstruct b2 when the colors, and the numbers of a2 and b1 are given. The conditions that should be satisfied: the remainder mod 3 is given, and the difference with both numbers, of a2 and b1, are given. If the numbers of a2 and b1 are equal, then the

two conditions are the same, so the whole reconstruction is the same as before, so there's nothing new to prove here.
So it remains to consider the case when the numbers a2 and b1 are different. Then the difference between them is 2, since they differ by 1 from the same number. So they have two different colors, so there is only one choice for the color of b2, to be different from the both colors. The average of a2 and b1 is the only number that differs by 1 from both and it is of color different from both, so that is the only possible reconstruction of that cell.
So, reconstruction is in both cases feasible and unique. QED.

2. Let ABCD be a convex planar cyclic quadrilateral (מרובע קמור חסום) and P a point in space. Show that $PD^2 \cdot S_{ABC} + PB^2 \cdot S_{ACD} = PA^2 \cdot S_{BCD} + PC^2 \cdot S_{ABD}$.

Solution. Choose a Cartesian coordinate system such that P is the origin and plan ABCD corresponds is $z = k$. The coordinates of our points are $A(x_a, y_a, k), B(x_b, y_b, k), C(x_c, y_c, k), D(x_d, y_d, k)$.
Since the points belong to the same circle, the pairs $(x_a, y_a), (x_b, y_b), (x_c, y_c), (x_d, y_d)$ all satisfy the equation:

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0.$$

Consider the matrix

$$\begin{pmatrix} x_a & y_a & 1 & x_a^2 + y_a^2 + k^2 \\ x_b & y_b & 1 & x_b^2 + y_b^2 + k^2 \\ x_c & y_c & 1 & x_c^2 + y_c^2 + k^2 \\ x_d & y_d & 1 & x_d^2 + y_d^2 + k^2 \end{pmatrix} = \begin{pmatrix} x_a & y_a & 1 & PA^2 \\ x_b & y_b & 1 & PB^2 \\ x_c & y_c & 1 & PC^2 \\ x_d & y_d & 1 & PD^2 \end{pmatrix}$$

The vector $\begin{pmatrix} \alpha \\ \beta \\ \gamma - k^2 \\ 1 \end{pmatrix}$ belongs to its kernel, so it is degenerate either way.

So determinant is 0. Decompose the determinant along the last column and You get the required identity (multiplied by 2), since minors are twice the areas of triangles.

Second solution (from the work of Dan Carmon). Perform inversion with center at P (and radius 1). Points A, B, C, D will go to points A', B', C', D'.
It is well-known that inversion turns generic spheres (i. e. spheres or planes) to generic spheres. So, intersection of generic spheres (which are circles or lines) are

turned into intersections of generic spheres. So, A'B'C'D' is still cyclic (or collinear). WLOG, it is cyclic: if we prove the formula in the case when P is not on the circle, the degenerate case follows by continuity of the both sides of the identity.

We shall use the famous formula for distance after inversion: A'B' = AB/(PA·PB). If you don't know it, please prove it (hint: similarity of triangles).

All triangles ABC, ABD, ACD, BCD are inscribed in the same circle of radius R, so their area might be computed as S_{ABC} = AB·BC·CA/(4R).

Substitute all areas with that formula to the identity we need to prove, and multiply by 4R. We get an expression with lengths only, without areas:

$$PD^2 \cdot AB \cdot BC \cdot CA + PB^2 \cdot AC \cdot CD \cdot DA = PA^2 \cdot BC \cdot CD \cdot DB + PC^2 \cdot AB \cdot BD \cdot DA$$

Divide by $PA^2 \cdot PB^2 \cdot PC^2 \cdot PD^2$. You get, by formula of distance after inversion,

$$A'B' \cdot B'C' \cdot C'A' + A'C' \cdot C'D' \cdot D'A' = B'C' \cdot C'D' \cdot D'B' + A'B' \cdot B'D' \cdot D'A'$$

If A'B'C'D' is circumscribed, we divide by 4R' where R' is the radius of A'B'C'D' circumcircle, we get

$$S_{A'B'C'} + S_{A'C'D'} = S_{B'C'D'} + S_{A'B'D'}$$

That is obvious, since both are equal to the area of quadrilateral A'B'C'D'.

3. It is given that $\sum_{i=1}^{\infty} x_i$ converges, and $\{x_i\}$ is a sequence of real numbers.

Can we claim that $\sum_{i=1}^{\infty} \sin(x_i)$ converges?

Answer. No.

Solution. For any *y*, consider triple: $2y, -y, -y$.

Apply sine to all numbers in triple: $\sin(2y), \sin(-y), \sin(-y)$.

Denote $f(y) = \sin(2y) + \sin(-y) + \sin(-y) = 2\sin(y)(\cos(y) - 1)$.

It is nonzero when *y* is sufficiently close to 0.

If for some *y* we repeat $2^n \left\lceil \frac{1}{|f(y)|} \right\rceil$ triples of that kind in the series, then sum of the corresponding interval in series $\sum x_i$ is 0, and the sum of corresponding interval in the $\sum \sin(x_i)$ has absolute value above 2^n.

So, take intervals of $2^n\left\lceil \frac{1}{\left|f(1/n)\right|} \right\rceil$ triples constructed from $y = \pm\frac{1}{n}$.

The series $\sum x_i$ will be converging, since each triple gives 0, and elements tend to zero, so the estimate on the absolute value of every tail of this series is 4ε, if ε is the estimate on absolute value.

At the same time, $\sum \sin(x_i)$ diverges, since it consists of intervals, and contribution of each interval is above 2^n by absolute value.

4. Suppose A is an m×n matrix and B is an n×m matrix. Prove that the set of nonzero eigenvalues of AB coincides with the set of nonzero eigenvalues of BA.

First solution. By symmetry, WLOG, $m \le n$. Let A' and B' be n×n matrixes created from A and B by adding 0 rows below and columns on the right. Then B'A' = BA, and A'B' is a block matrix, first block is AB, second block is 0-matrix.
Anyway, eigenvalues of BA and of B'A' are the same, and nonzero eigenvalues of AB and of A'B' are the same, so from now on WLOG we may assume that the matrixes A and B were square matrixes from the beginning, i. e. $m = n$.
If B is invertible, then AB = $B(AB)B^{-1}$ so the matrixes AB and BA are similar, so their eigenvalues coincide.

Lemma. The set of invertible matrixes is dense in the set of matrixes, in other words for any non-invertible matrix B there is a sequence of matrices $\{B_n\}$ such that $B_n \xrightarrow[n\to\infty]{} B$.

This lemma allows to extend the claim from invertible matrixes to non-invertible. Indeed, for any non-invertible B we have a sequence of invertible matrixes $B_n \xrightarrow[n\to\infty]{} B$. For any element of this sequence, AB_n is similar to B_nA, in particular AB_n and B_nA have the same characteristic polynomial. The coefficients of the characteristic polynomial are polynomials in matrix elements, so characteristic polynomials of AB and of BA are limits of characteristic polynomials of AB_n and B_nA respectively, so they are equal. Since AB and BA have the same characteristic polynomials, their polynomials should have the same sets of nonzero roots, QED.

It remains to prove the lemma.

Proof of lemma. The lemma is a direct result of the combination of 3 facts:

a. The set of invertible matrixes is non-empty.
b. Non-invertible matrixes are defined by a polynomial in coefficients.
c. In $\mathbb{R}^N$, the set of non-zeroes of given polynomial is either dense or empty.

The fact a' is obvious, the b' is also obvious, so it remains to prove c'.

So, assume a polynomial $p(x_1, x_2, ..., x_N)$ has nonzero value at point $(g_1, g_2, ..., g_N)$ and we need to find a sequence of such points converging to a given point $(x_1, x_2, ..., x_N)$. Draw a line in space passing via this 2 points, in parametric form $(x_1 + tk_1, x_2 + tk_2, ..., x_N + tk_n)$, here t is the parameter. Consider our polynomial restricted to that line $q(t) = p(x_1 + tk_1, x_2 + tk_2, ..., x_N + tk_n)$, it is a polynomial in t. The polynomial q is nonzero for at least one value (when the line goes via th point $(g_1, g_2, ..., g_N)$, so it is a nonzero polynomial of one variable and it has only finite number of roots, so the non-zeroes are dense in this line and we can construct the sequence. QED.

Second solution. Assume λ is not an eigenvalue of AB. Then AB – λI is an invertible matrix and its inverse is C, i.e. I = (AB – λI)C = ABC – λC.

Consider matrix BCA.

$$(BA - \lambda I)(BCA) = B(ABC)A - \lambda BCA = B(\lambda C + I)A - \lambda BCA =$$
$$= \lambda BCA + BA - \lambda BCA = BA$$

Therefore (BA – λI)(BCA – I) = BA – BA + λI = λI.

So, if λ is nonzero, then (BCA – I)/λ is the inverse matrix of BA – λI.

To summarize: if a nonzero λ is not an eigenvalue of BA, then it is also not an eigenvalue of AB. Vice versa is also true by symmetry.

Third solution (from the work of Gal Dor). Let $m(x)$, $n(x)$ be the minimal polynomials of AB, BA respectively.

So, by definition $m(AB) = a_k(AB)^k + \ldots + a_2(AB)^2 + a_1 AB + a_0 I = 0$

Multiply by B from the left and by A from the right. You get:

$$a_k(BA)^{k+1} + \ldots + a_2(BA)^3 + a_1(BA)^2 + a_0 BA = 0$$

This is what happens when you apply polynomial $xm(x)$ to BA.

Any polynomial that nullifies BA is divisible by $n(x)$.

Therefore $xm(x)$ is divisible by $n(x)$.

For the same reason $xn(x)$ is divisible by $m(x)$.

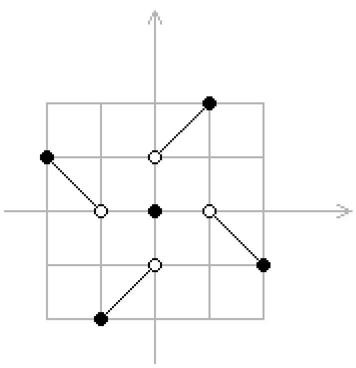
Hence $m(x)$ and $n(x)$ have the same nonzero roots, QED

(and the same multiplicities, and multiplicity of 0 differs by 1 at most).

Fourth solution (from the work of Ilya Gringlaz). Assume λ is a nonzero eigenvalue of AB, so for a curtain vector v we have ABv = λv.
Notice that Bv is nonzero, otherwise ABv would be 0 and not λv.
But BABv = Bλv = λBv, so vector Bv is nonzero and it gets multiplied by λ when we multiply it by BA, so BA has λ as an eigenvalues with eigenvector Bv.

5. (a) Find a function defined on closed interval [-1,1], which has only finite number of discontinuity point, such that its graph is invariant under rotation by the right angle around the origin.
(b) Prove that there is no function on open interval (-1,1) which satisfies the same conditions.

Solution. (a) One of the possible examples:



$$f(x)=\begin{cases} -\tfrac{1}{2}-x & x\in[-1,-\tfrac{1}{2}) \\ x-\tfrac{1}{2} & x\in[-\tfrac{1}{2},0) \\ 0 & x=0 \\ x+\tfrac{1}{2} & x\in(0,\tfrac{1}{2}] \\ \tfrac{1}{2}-x & x\in(\tfrac{1}{2},1] \end{cases}$$

(b) First solution. The graph consists of the finite number of continuous intervals, open, closed and half-open (the isolated points will be considered as very short closed intervals), because there is only finite number of discontinuity points.
On each interval function is strictly monotone, since if some value is accepted twice then after 90° rotation we would see 2 values for the same x.
The continuity interval of the graph containing 0 is an isolated point. Would it be longer, than for x sufficiently close to 0 from one side the sign of $f(x)$ would be the same, and that would contradict invariance with respect to rotation by the right angle.
Except for isolated point at 0, no continuity interval of the graph will go to itself after rotation by 180° around the origin. Indeed, if it would, than it would contain 0, and we proved it is impossible.
Also, except isolated point at 0, no continuity interval of the graph would go to itself after rotation by 90° around the origin, since then it would also go to itself after two rotations of that kind, and that is impossible.

Let **S** be the set of all continuity intervals of the graph except the isolated point at 0. Rotation by 90° around the origin divides **S** into orbits of four. Consider two ends of each element of **S**: each of them can be either open or closed. Consider the total number of open ends in **S** minus total number of closed ends.

Each element of S contributes 2, -2 or 0 to this quantity, so each orbit of four contributes something divisible by 8. On the other hand, each non-integer discontinuity point gives 1 open end and 1 close end which cancel out, and at 0, 1, -1 we have 4 open ends, so the total quantity is 4.
Contradiction: 4 is not divisible by 8.

Second solution. Like in the first solution, we explain that there are orbits of four and one separate point. Union of all continuity intervals is the domain.
Now we count Euler characteristic. Euler characteristic is additive. Euler characteristic of a point is 1, and of an open interval is -1.
So, Euler characteristic of the domain should be 4k+1, and it isn't.

Remark. Another way to formulate the main condition of this problem, about the rotation by right angle, is $f(f(x)) = -x$.