Second stage of Israeli students competition, 2009.

1. Which is bigger: arctan(e) or $\frac{\pi}{4}+\frac{1}{2}$?
Calculator is not allowed.

First solution. It is the same as to ask what is greater $\arctan(e)-\frac{\pi}{4}$ or $\frac{1}{2}$.

$$\arctan(e)-\frac{\pi}{4}=\arctan(e)-\arctan(1)=\int_1^e \arctan'(x)\,dx=$$

$$=\int_1^e \frac{dx}{1+x^2}<\int_1^e \frac{dx}{2x}=\frac{1}{2}\ln(x)\Big|_1^e=\frac{1-0}{2}=\frac{1}{2}$$

That is because for all $x > 1$ we have $1 + x^2 > 2x$ since it is the same as $(x - 1)^2 > 0$.

Therefore, arctan(e) < $\frac{\pi}{4}+\frac{1}{2}$.

Second solution. It is the same as to compare e versus $\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)$, since arctan is monotonously increasing.

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{1}{2}\right)}=\frac{1+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{1}{2}\right)}$$

The expression $\frac{1+x}{1-x}$ is monotonously increasing for $0 < x < 1$. Indeed, when x is increasing then $1-x$ is decreasing so $\frac{1}{1-x}$ is increasing, and $1+x$ is increasing too, so their product is increasing. But $\tan\left(\frac{1}{2}\right)>\frac{1}{2}$ hence

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{1+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{1}{2}\right)}>\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3>e$$

QED.

2. Prove that $\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+...+\frac{1}{3n+1}$ is non-integer for any n.

Solution. Consider 2^N, the greatest power of 2 which appears in the sequence of denominators 4, 7, 10, … , $3n + 1$. The question is, whether this sequence of denominators contains other numbers divisible by this power of two.
If not, then multiplying by $2^{N-1}\cdot 1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot(4n + 1)$ will turn all the summands except one into integer numbers, so number will be non-integer even after multiplying by such a large integer number, so in this case the problem is solved.
If yes, then the sequence of denominators contains another number of the form $k2^N$. Here k cannot be 2 or 3 because multiplying by k turns number of the form $3m+1$ into another number of the form $3m+1$, so k is 4 or greater. But $4\cdot 2^N$ is a greater power of 2, and it turns out to be in the sequence of denominators.
This is a contradiction, since we have chosen the greatest power of 2.

3. A triangle is contained by an 11-dimensional unit cube inside P^{11}. What is the maximal possible perimeter of that triangle?

Answer. $\sqrt{7}+\sqrt{7}+\sqrt{8}=2\left(\sqrt{7}+\sqrt{2}\right)$

Solution.
Lemma 1. The perimeter will be the greatest, if the vertices of triangle are the vertices of the cube.

It follows directly from:

Lemma 2. If two points are fixed, than the third point giving maximal sum of distances from the first two is a vertex of the cube.

Proof. Sum of distances from two given points is a convex function. That happens because each of those is a convex function, and sum of the convex function is a convex function.

Reminder. A convex function is a function, for which above-the-graph domain is convex (above-the-graph domain is $\{ (x , y) \mid f(x) < y \}$, here x may be a vector). Considered on a closed interval, convex function has maximum in one of the ends. So, considered on the bounded polygon, convex function has maximal value at on of its vertices. Distance function is convex since above-the-graph is a cone over a ball, which is a convex body.

Now, back to 11 dimensions. Because of lemma 1, the problem degenerates into a combinatorial problem. Instead of trying to find 3 points, we have 3 sequences of zeroes and ones. The distance between two points in each coordinate is 0 or 1 also, so the distance is square root of number of differences.
Between all 3 points in a given coordinate there are not more than 2 differences. Therefore, in all 11 coordinates, there are no more than 22 differences. So, if numbers of differences between 3 coordinate sequences are *K*, *M*, *N* then the perimeter is $\sqrt{K}+\sqrt{M}+\sqrt{N}$, which should be maximal while $K+M+N \leq 22$.

Lemma 3. If $N > K+1$, then $\sqrt{K}+\sqrt{N}<\sqrt{K+1}+\sqrt{N-1}$. (Actually, this kind of lemma is true for any concave function, not just for square root)

Proof of lemma 3. Reformulate it:

$$\sqrt{N}-\sqrt{N-1}<\sqrt{K+1}-\sqrt{K}$$

Multiply and divide by adjoint:

$$\frac{1}{\sqrt{N}+\sqrt{N-1}}<\frac{1}{\sqrt{K+1}+\sqrt{K}}$$
$$\sqrt{N}+\sqrt{N-1}>\sqrt{K+1}+\sqrt{K}$$

So, $\sqrt{K}+\sqrt{M}+\sqrt{N}$ the number will be maximal none among *K*, *M*, *N* differ by more than 1. Of course, we may also assume $K+M+N = 22$, otherwise adding 1 to one of the numbers will improve $\sqrt{K}+\sqrt{M}+\sqrt{N}$.
So, all *K*, *M*, *N* are equal to either *L* or *L*+1 and $22 = K+M+N = 3L + R$, where *R* is 0, 1, 2, or 3. So, $L = 7$, R = 1, and *K*, *M*, *N* are 7, 7, 8 in some order.

Of course, after seeing that 7, 7, 8 is the best under algebraic restriction we got from the cube, we have to check that these lengths are attainable in our cube.
For example:
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
(1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0)
(0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1)
So, the greatest possible perimeter is $\sqrt{7}+\sqrt{7}+\sqrt{8}=2\left(\sqrt{7}+\sqrt{2}\right)$.

4. Can a polynomial with rational coefficients have $-\sqrt{2}$ as its minimal value?

First solution. Let us try $p'(x)=k\left(x^2-2\right)\left(x-a\right)=x^3-ax^2-2x+2a$.

$$p(x)=k\left(\frac{x^4}{4}-\frac{ax^3}{3}-x^2+2ax\right)+c.$$

The extremal points are $\pm\sqrt{2}$, a, so when we substitute them into $p(x)$ we have good chances to get something with $\pm\sqrt{2}$. Of course, if $k > 0$, then the middle extremum is a maximum, and the other two are minima.

$$p\left(\pm\sqrt{2}\right)=k\left(1\mp\frac{2a\sqrt{2}}{3}-2\pm 2a\sqrt{2}\right)+c=k\left(-1\pm 2a\sqrt{2}\frac{2}{3}\right)+c$$

Choose $a=\frac{3}{4}$. Then that will be the middle extremum.

The local minima are at $\pm\sqrt{2}$, and the global minimal value is the least between $p\left(\pm\sqrt{2}\right)=k\left(-1\pm\sqrt{2}\right)+c$, which is $p\left(-\sqrt{2}\right)=k\left(-1-\sqrt{2}\right)+c$. Take $k = c = 1$ and You get a polynomial with rational coefficient satisfying all conditions.

Second solution. Consider $q(x)=\left(x^2-2\right)^2=x^4-4x^2+4$.

It is zero at $\pm\sqrt{2}$, and positive elsewhere.

Now consider polynomial satisfying $r'(x)=\frac{3}{4}\left(x^2-2\right)$, $r(x)=\frac{x^3}{4}-\frac{3x}{2}$.

That polynomial has extrema at $\pm\sqrt{2}$, a local maximum at $-\sqrt{2}$ and a local minimum at $\sqrt{2}$. The coefficient was chosen so that $r\left(\sqrt{2}\right)=\frac{2\sqrt{2}}{4}-\frac{3\sqrt{2}}{2}=-\sqrt{2}$.

Now consider a polynomial $p(x)=r(x)+Aq(x)$, where A is a positive number.

It has local minimum with value $-\sqrt{2}$ at $\sqrt{2}$, and positive value of $\sqrt{2}$ at $-\sqrt{2}$.

The values at far negative numbers are positive, since x^4 is stronger than x^3.

If we enlarge A then values outside small neighbourhoods of $-\sqrt{2}$ and $\sqrt{2}$ become as big as we can wish, say positive. Since values near $-\sqrt{2}$ are also positive, the value at $\sqrt{2}$, which is $-\sqrt{2}$, becomes a global maximum.

5. Consider a shape consisting of a finite number of unit square cells.

We try to cover a board of $m\times n$ cells by equivalent copies of that shape, so that each cell of the board will be covered by the same number of layers.

Prove that it is impossible if and only if we can write a real number in each cell of the board, in such a way that the sum of all those numbers will be strictly negative, while a sum that can be covered by the given shape is strictly positive (wherever we place it on the board).

Solution. Consider an mn-dimensional linear space of all tables with real values in the cells. For each cell we can take a coordinate unit vector in that space, and a scalar product between that vector and the vector of that table will give the value in that cell.
To each subset of cells we match the sum of the unit vectors. The scalar product with that vector will give sum of numbers in the corresponding set.
Suppose that we have a set of some shapes inside the board, and we construct a vector corresponding to each. To tile the board in $k > 0$ layers is the same as to find integer nonnegative coefficients such the linear combination will be $(k, k, k, \dots, k)$. Which is the same as to express $(1, 1, 1, \dots, 1)$ as a linear combination of some of those vectors with positive rational coefficients.
The rest of it follows from two lemmas:

Lemma 1. Consider vectors with rational coordinates $v_1, v_2, \dots, v_q$ and v.
If v is a linear combination of $v_1, v_2, \dots, v_q$ with positive real coefficients then it is a linear combination of $v_1, v_2, \dots, v_q$ with positive rational coefficients.

Lemma 2. Consider vectors with nonnegative coordinates $v_1, v_2, \dots, v_q$ and v.
If v is not a linear combination of $v_1, v_2, \dots, v_q$ with positive real coefficients then there exists a vector u such that $(u, v) < 0$ and $(u, v_i) > 0$ for $i = 1, 2, \dots, q$.

From the first lemma we see, that if there is no tiling, then the vectors corresponding to our shapes don't generate the vector of the whole board as a linear combination with nonnegative coefficients. From the second lemma we see that in such case there is a table which has negative sum over all the cells and positive sum over each shape. That proves the problem in a non-trivial direction. (The other direction is obvious: if such a table exists, then the tiling doesn't, since the sum in the cells of that tiling should be negative, but it will be positive.)
So, it remains to prove the lemmas.

Proof of lemma 1. Solution of the system of linear equations, which is written by one vectorial equation $x_1v_1 + x_2v_2 + \dots + x_qv_q = v$ is a shifted linear subspace in the q-dimensional space. So, if it exists (and it is given it has a positive real solution), it can be solved by Gauss method and we shall have an answer:
$(x_1, x_2, \dots, x_q) = u + y_1u_1 + y_2u_2 + \dots + y_tu_t$, where $u, u_1, u_2, \dots, u_t$ are some q-dimensional vectors, and $y_1, y_2, \dots, y_t$ are arbitrary real numbers.

Since Gauss method is an algebraic procedure, all the coordinates of u, u_1, … , u_t will be rational.
It is given that for some values of y_1, y_2, … , y_t all the coordinates will be positive real numbers, so they will also be positive if we change y_1, y_2, … , y_t by sufficiently small numbers, since linear functions are continuous. But in any neighborhood of each real number there is a rational number. So we can shift coordinates slightly so that x_1, x_2, … , x_q will remain positive and y_1, y_2, … , y_t will be rational, but then all x's will also be rational since they are algebraic expressions in y's and coordinates of u's.

Proof of lemma 2. Let S be a hyperplane defined by the equation:
Sum of all coordinates = 1.
Positive linear combinations of v_1, v_2, … , v_q cut S along a convex body C.
This convex body is bounded, since it is inside a simplex, whose vertices are coordinate vectors, because the coordinates are positive (a simplex is a multidimensional generalization of triangle).
A ray generated by vector v cuts S at a point P which is not in C.
We shall prove that there is a sub-hyperplane T in S, such that P is on one side of T and P is on one side, and C is on the other side.
From that it will follow that a hyperplane, passing through T and 0, such v will be on one side, and v_1, v_2, … , v_q on the other side, and the equation defining that hyperplane will have one sign on v_1, v_2, … , v_q and another sign on v.

So, it remains to prove the following statement inside hyperplane S, which is Euclidean space by itself:
Lemma 3. Let C be a compact convex body in a Euclidean space, and P a point outside C, then there exists a hyperplane T that defines that C is which separate P from C.

Proof of lemma 3. Let Q be the point of C closest to P (it exists since C is compact).
Let T be a perpendicular bisector to interval QP. (Perpendicular bisector is a hyperplane cutting the interval perpendicularly in the middle, it is also the set of all points which are at the same distance from both ends).
We shall prove that T separates P from C. Suppose not: there is a point R in C either on T itself or on the same side of T as P. The whole interval QR is in C, since P is convex. But the angle PQR is acute. So, if we start going by QR from Q to R we get closer to P, at least at first. But Q is the point of C that is closest to P, that is a contradiction. QED.
Remark. We don't really need the convex set to be compact for lemma 3, enough to require that it is closed. Infinite-dimensional version of lemma 3 is called Hahn-

Banach theorem, and it is considered one of the central theorems of functional analysis.