Targil 9 - unique factorization.

Reminders:
(a) A ***ring*** is a set of numbers with 3 arithmetic operation: +, –, ·, and some obvious axioms (for lists of axioms, use google/wikipedia).

(b) $\mathbb{Z}[\alpha]$ = the ring generated by integers and α.

(c) ***Unit***s of the ring are elements invertible under multiplication (such a that there exists b such that $ab = 1$).
(d) Element a is ***irreducible***, if for any decomposition $a = bc$ either b or c is a unit (but not both).
(e) ***Irreducible factorization*** of some element of a ring is a representation of it as a product of irreducible elements.
(f) A ring has ***unique factorization*** property, if any two irreducible factorizations of any element differ only by permutation of factors and multiplication of factors by units.

1. Consider ring $\mathbb{Z}\left[i\sqrt{d}\,\right]$, where d is a positive integer.

(a) Prove that if $d < 3$ there is unique factorization property.
(b) Prove that if $d \geq 3$ there is no unique factorization property.

2.* Find all integer solutions of equations:
(a) $x^2 + 4 = y^3$
(b) $x^2 + 2 = y^3$

3. Represent 1234321 and 123454321 as $a^2 + b^2$, where a, b are positive integers (using computer here is the moral equivalent of driving the morning run, and has the same effect but on a different group of muscles☺).

4. (a) How many ways are there to represent $2^k\, p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} \cdot q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_m^{j_m}$ as a sum of 2 integer squares, if p_i are prime numbers of type $4a + 1$ and q_j are prime numbers of type $4b + 3$?
(b)* For R > 0, show that number if integer points in the disc $\{ x^2 + y^2 \leq R \}$ is

$$1+4\left([R]-\left[\frac{R}{3}\right]+\left[\frac{R}{5}\right]-\left[\frac{R}{7}\right]+\left[\frac{R}{9}\right]-\left[\frac{R}{11}\right]+\left[\frac{R}{13}\right]-\ldots\right).$$

(c)* Consider "triangular" lattice, formed by points of $\mathbb{Z}[\omega]$, where $\omega \neq 1,\ \omega^3 = 1$.
Show that number of points of this lattice in the disc $\{ x^2 + y^2 \leq R \}$ is

$$1+6\left([R]-\left[\frac{R}{2}\right]+\left[\frac{R}{4}\right]-\left[\frac{R}{5}\right]+\left[\frac{R}{7}\right]-\left[\frac{R}{8}\right]+\left[\frac{R}{10}\right]-\left[\frac{R}{11}\right]+\ldots\right).$$

5. Suppose $n > m > 0$ are integers, $\phi = \arctan(m/n)$, prove $\left\{\frac{k\phi}{\pi}\right\} > \frac{1}{\pi}\left(\frac{1}{\sqrt{m^2+n^2}}\right)^k$.

($\{x\}$ denotes fractional part of x, which is a number in [0,1) equivalent to x mod 1)