

Targil 4 – parity and divisibility.

1. We are given a herd 2009 cows. For each cow, if it is taken aside, others can be divided into two sub-herds of 1004 cows and equal total weight. Prove that all the cows have the same weight.

Solution. Let us fix a partition into two halves for each 2008 cows. This choice of partitions will be called *configuration*. The condition that both halves of the partition will be of the same weight specifies a linear equation for the weights of cows, $x_1, x_2, \dots, x_{2009}$. So the complete configuration specifies a system of 2009 linear equations. Those equations might be linearly dependent. However, with Gauss method, we can solve it: we can take several unknowns as parameters and obtain get the other unknowns as linear expressions in those parameters. All coefficients in those expressions will be rational, since Gauss method is an algebraic procedure.

So, for each configuration, and for each possible set of weights of cows, we can find ε -close rational weights of cows which satisfy all the condition with the same configuration. So, without loss of generality, we can assume that if not all cows will have the same weight, then they have rational weights. So, by multiplying weight of each cow by the denominator (or, alternatively, by choosing appropriate measurement units, since multiplying weight of a cow by a large number might not have physical meaning), we can assume, WLOG, that all cows have natural weights.

Out of all solutions with natural weights of cows, with not all cows of the same weight, take the minimal one, with minimal total weight. Then all weights cannot be simultaneously even, since otherwise we would be able to divide by 2. Also, all weight all weights cannot be simultaneously odd, otherwise we could add 1 and divide by 2 and get smaller solution, unless all cows are of weight 1. But all cows have to be of the same parity, since if we take any cow apart, we remain with the herd of even weight (since it can be divided into two equal parts). That is a contradiction. QED.

2**. (a) A square is divided into N triangles of equal area. Prove that N is even.
(b) Generalize it for higher dimensions (a cube is divided into simplexes).

Alexey believes he had a very nice proof of this, which we cannot reconstruct. Several ideas he remembers from that prove:

- (1) Assume first coordinates are rational and arrive to a contradiction with that.
- (2) If all coordinates are rational, multiply everything by the common denominator and make them integer. If common denominator was odd, it turns out that vertices of each triangle are on one line mod 2. Because an expression in coordinates with vector products is equal twice the area, so if number of triangles is odd those expressions turn out to be even. But the vertices of the square are not on one line mod 2.
- (3) If the denominator is even, shift all odd coordinates by 1 mod 2, so that all conditions will still hold, and then divide all coordinates by 2.
- (4) Though we cannot assume all coordinates are rational, we can assume all coordinates are algebraic. Algebraic numbers are similar to rational in some ways, and the proof for rational numbers can be modified to work with algebraic numbers.

3. On infinite empty chessboard, a rectangle of $m \times n$ pieces is placed. One type of operation is allowed: a piece can jump above the piece in adjacent cell to the next cell after it, which should be free, and then the piece above which it jumped is removed.

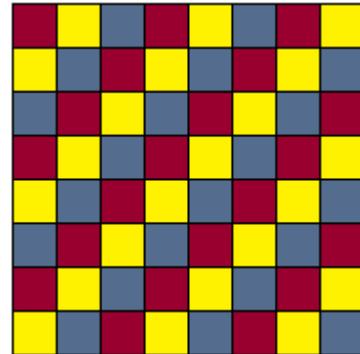
By adjacent cells we mean cells with common side.

The purpose of the game is to leave only one piece on the board.

For which m, n is it possible?

Answer. When both m, n are >1 and not divisible by 3, or when they are 1 and 2.

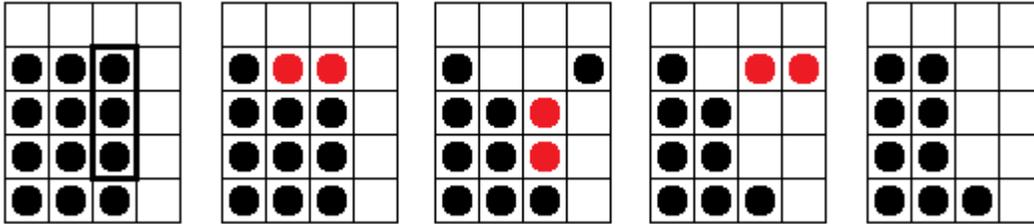
Solution. We can paint the board into 3 colors, like on the picture. Each move is equivalent to inversion of 1×3 rectangle, so number of pieces on each color will change by 1, so parity of pieces on each color will change. If one side of rectangle is divisible by 3, then



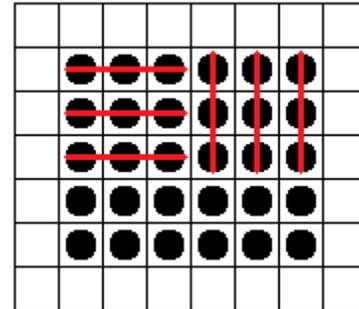
in the beginning there is the same number of pieces on each color, so on any stage of the game, we shall have either odd number on all colors or even number of all colors, so we won't arrive to 0, 0, 1 and so we shall never remain with just one piece.

If one side of rectangle is 1 we are trapped in just one line. We have just one sequence, so we can make a move on each edge outside, and after each move we shall still have just one sequence and a few separated discs, and we shall not be able

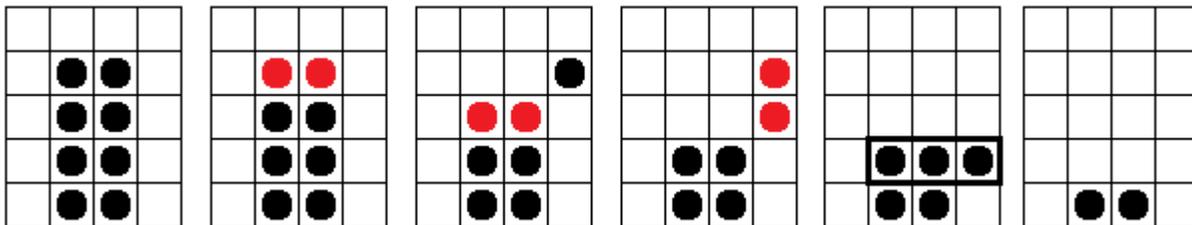
Now for the good case. Here is a way to eliminate a rectangle 1×3 near the corner:



Using this elimination, we can turn a rectangle $m \times n$ into rectangle $m \times (n-3)$ for $m \geq 3$, $n > 3$: eliminate top 3 discs in each column one by one, from right to left until only 3 columns remain, then eliminate top 3 discs with one move 3 times.



We can also turn a rectangle $m \times n$ into rectangle $m \times (n-3)$ for $m = 2$, $n > 3$, using 3 moves and elimination of a triple:



Also, for $m, n > 3$ we can turn rectangle $m \times n$ into rectangle $(m-3) \times (n-3)$ in a method, similar to the described above: from right to left, eliminate top 3 discs in each column until only 3 remain, then from top to bottom, eliminate left 3 discs in each row until only 3 remain, then from left to right, eliminate 3 leftmost discs 3 times.

After several procedures of that kind (reducing the biggest side of the rectangle each time if the difference between sides is bigger than one, and if sides are almost equal reducing them simultaneously), if m and n are not divisible by 3, we shall get either rectangle 2×2 , or 2×1 , or 1×1 . The case 2×2 is reduced to 2×1 by 2 horizontal moves in the same direction. The case of 2×1 is reduced to 1×1 by 1 move.

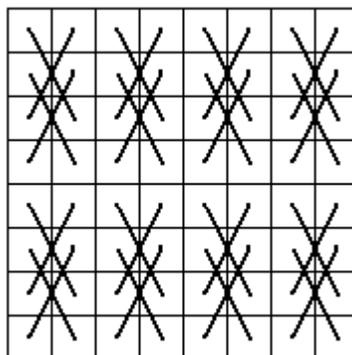
4. Two players play a game on the standard empty chessboard.

They have a chess knight (horse). The first player places it on the chessboard at any cell he wishes, then the second makes a legal move with the knight, then the first makes a legal move and so on. In addition to standard chess rules, the knight is forbidden to step on the same cell twice.

The player that can't make a move in his turn loses. Who of the two players has a chance to win?

Answer. The second.

Solution. Divide the cells into pairs, connected by the move of the knight (for example, like in the picture). Each time the first player takes one of the cells of a certain pair, the second will take another pair.



5*. Let T be the set of all numbers of the form m^n , where $m > 1$ and $n > 1$ are integer. Compute $\sum_{t \in T} \frac{1}{t-1}$.

(Since T is a set, a number which can be represented both as m^n and a^b is counted only once.)

Remark. This problem was proposed to us by Dan, he also wrote down both solutions below.

First Solution Denote for any number $t \in T$, $c_1(t) = \#\{(m, n) \mid m, n \geq 1, m^n = t\}$ the number of ways to represent t as a power. Also denote by $c_2(t) = \#\{(m, n) \mid m, n \geq 2, m^n = t\}$ the number of ways to represent t non-trivially as a power, and $c_3(t) = \#\{(m, n) \mid m \in T, n \geq 1, m^n = t\}$ the number of ways to represent t as a power of a perfect power.

It is easy to verify that for any $t \in T$, $c_2(t) = c_3(t) = c_1(t) - 1$.

We now have, using summations of geometric and telescopic series:

$$\begin{aligned}
\sum_{t \in T} \frac{1}{t-1} &= \sum_{t \in T} \sum_{n=1}^{\infty} \frac{1}{t^n} \stackrel{s=t^n}{=} \sum_{s \in T} \sum_{(t,n): t \in T, n \geq 1, t^n = s} \frac{1}{s} = \sum_{s \in T} \frac{c_3(s)}{s} = \\
&= \sum_{t \in T} \frac{c_2(t)}{t} = \sum_{t \in T} \sum_{(m,n): m, n \geq 2, m^n = t} \frac{1}{t} = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{m^n} = \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \\
&= \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1
\end{aligned}$$

Second Solution (Due to Euler and Goldbach)

Denote by $S = \mathbb{N} \setminus T \setminus \{0,1\} = \{n \geq 2 \mid n \notin T\}$ the set of all numbers greater than 1

which are not perfect powers. Let $\mathbf{x} = \sum_{n=1}^{\infty} \frac{1}{n}$ denote the harmonic series. Then we

my rearrange and write $\mathbf{x} = \sum_{t \in T} \frac{1}{t-1} + \sum_{s \in S} \frac{1}{s-1}$, as $T \cup S = \{2,3,4,\dots\}$. On the other

hand, we have the property that any number $n \geq 2$ can be expressed uniquely in the form $n = s^m$ where $s \in S$ is a non power and $m \geq 1$ some natural. Hence by using

the geometric series we get $\sum_{s \in S} \frac{1}{s-1} = \sum_{s \in S} \sum_{m=1}^{\infty} \frac{1}{s^m} = \sum_{n=2}^{\infty} \frac{1}{n} = \mathbf{x} - 1$, from which we can

derive $\sum_{t \in T} \frac{1}{t-1} = \mathbf{x} - \sum_{s \in S} \frac{1}{s-1} = \mathbf{x} - (\mathbf{x} - 1) = 1$.

Note that this proof is far from rigorous: The value of the harmonic series \mathbf{x} is well known to be infinite, and so it is not quite necessary that rearrangements as those we did by adding and subtracting \mathbf{x} , as well as manipulating the terms of the divergent $\sum_{s \in S} \frac{1}{s-1}$, will really yield the correct answer. However, the proof can be

modified (with some work) to a rigorous proof, as is done in [this article](#).

(<http://www.recercat.net/bitstream/2072/920/1/776.pdf>)