Targil 3 - functions.

1. Consider a function $f(x) = x - \frac{1}{x-a_1} - \frac{1}{x-a_2} - \ldots - \frac{1}{x-a_n}$, where $a_1, \ldots a_n$ are some real constants. Compute the total length of $f^{-1}([a,b])$.
(Here $[a,b]$ is an interval between a and b, $f^{-1}(set)$ denotes the inverse image of that *set* under f, that is all points that are sent by f to that set, and total length of several intervals is the sum of their lengths).

2. Let $f(x) = 2x(1-x)$, for $x \in \mathbb{R}$. Define $f_n(x) = f(\ldots f(f(x))\ldots)$, where f is applied n times.

(a) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

(b) Compute $\int_0^1 f_n(x)\,dx$ for all natural n.

3. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and $f(x+y) \geq f(x) + y \cdot f(f(x))$ for all $x, y \in \mathbb{R}$.

4. Prove that for every continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$,

$$\int_0^1 \int_0^1 (f(x,y))^2\,dxdy + \left(\int_0^1 \int_0^1 f(x,y)\,dxdy \right)^2 \geq \int_0^1 \left(\int_0^1 f(x,y)\,dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^2 dx$$

5*. Can a minimal value of a polynomial with rational coefficients be $\sqrt{2}$?

By minimal value here we mean the value at a point of global minimum.