# Targil 2 - some linear algebra.

**1.** Let R be a 3×3 matrix representing rotation of Euclidean space.
How to compute the angle of rotation? And the axis?

**Solution.** Angle can be computed as arccos((trace – 1)/2)
Indeed, if the axis of rotation is z axis of the space the matrix would take certain form, on the diagonal we would have two times cos(angle) and 1,
so trace = 2cos(angle) + 1.
But trace doesn't depend on the choice of basis, so this formula holds in any basis.
Axis is a solution of linear system $Ax = x$ and can be found by Gauss method.
Or, since that system is degenerate, and has one-dimensional solution, as a vector product of two linearly independent lines of the matrix.

And yes, for unit matrix axis of rotation is undefined.

**2.** Assume $\alpha \neq 0$ is a real number and $F$, $G$ are two linear maps (operators) on $\mathbb{R}^n$ such that $FG - GF = \alpha F$.
(a) Prove that $F^k G - GF^k = \alpha k F^k$.
(b) Prove that is $F^k = 0$ for certain $k$.

**Solution.** **(a)** Induction over k. Base of induction is given. Step from $k$ to $k$+1:

$$F^{k+1}G - GF^{k+1} = F^{k+1}G - FGF^K + FGF^K - GF^{k+1} =$$

$$= F\left(F^k G - GF^k\right) + \left(FG - GF\right)F^k =$$

$$= F\left(\alpha k F^k\right) + \left(\alpha F\right)F^k = \alpha k F^{k+1} + \alpha F^{k+1} = \alpha\left(k+1\right)F^{k+1}$$

**(b)** Consider a linear operator over the linear space of $n\times n$ matrices
$L(X) = XG - GX$.
If all $F^k$ are nonzero, than all of them are eigenvectors of that operator corresponding to different eigenvalues. But linear operator on finite-dimensional space can have only finite number of eigenvalues. QED.

**3.** (a) Is it true that for each couple of square matrices A, B, matrices AB, BA are similar?
(b) Is it true that A and $A^T$ are always similar?
(Reminder: matrices X and Y are similar iff $X = PYP^{-1}$ for some invertible P, that means, the matrices represent the same linear transformation in a certain basis.)

Answers (a) no (b) yes.

**Solution.** **(a)** For example

$$\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

**(b)** Every matrix A is similar to its Jordan form $J=PAP^{-1}$.
Then $A^T$ is similar to $J^T = (P^T)^{-1}A^TP^T$ (by the way, why $(A^T)^{-1} = (A^{-1})^T$ ?).
So, it is enough to prove, that a Jordan cell is similar to its transpose.
The similarity is performed by a matrix R, having 1s on the secondary coordinate and zeroes elsewhere. That is permutation matrix, it reverts the order of all coordinates, and $R^{-1} = R$. Multiplying by R from the left reverts the order of rows, and multiplying by R from the right reverts the order of columns, so conjugation by R rotates the matrix by $180^o$. That will bring a Jordan cell C to $C^T$.

**4*.** (a) Let $A_1A_2\ldots A_n$ be a regular polygon, O its center. For any point X, consider perpendiculars from X to the lines of sides of the polygons as vectors starting at X and ending on corresponding sides. Prove that sum of those vectors is $n$XO/2.
(b) In similar problem in a platonic solid of $n$ faces, the answer is $n$XO/3.

**Solution.** Everything depends linearly on X, so the formula in Cartesian coordinates should be MX + v, where M is a matrix, and v is a vector.
If the O is the origin, the result is O, since it is preserved by many rotations, and the only vector that is preserved by all those rotations is 0.
So the formula is linear, MX, multiplication by a certain matrix.
If X is on a perpendicular from O to a face, then rotation or symmetry that keeps this perpendicular line sends the polytope/polygon to itself, so X should be sent to $a$X, where $a$ is a constant. This constant is the same for all perpendiculars to faces, because of the symmetry. So, our matrix acts as multiplication by $a$ on all those vectors, and they span the whole space, so our matrix is $a$ times unit matrix.
But what is $a$?
$a$ = trace divided by dimension. Our linear transformation is some of projectional linear transformation – projecting vector to a line perpendicular to a certain face. Trace of each summand is 1, since trace doesn't depend on coordinates, and if the axis of projection would be a coordinate axis, matrix would have 2 in one corresponding diagonal call and 0 in all other sells.

So, we have summands, number of summands is equal to number of faces of the polytope, and trace of each is 1, so total trace = number of faces, hence $a$ = number of faces / dimension.

**5**.** Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients. Show that the determinant is a perfect square.

**Remark.** $\det A = \det A^T = (-1)^n \det A$, so it is nonzero (and non-obvious) only for even dimension.

**First solution.** Determinant is integer, so it is enough to prove the it is a square of rational number, then we shall know it is a square of integer. If we apply a certain permutation on rows and the same permutation on columns, matrix will remain anti-symmetric and will keep the same determinant.

So we may assume that unless the matrix consists of zeroes only, then cells near the left-top corner (1,2) and (2,1) are non-zero: one is $a$, another is $-a$. Then by adding linear combinations of first and second rows to all other rows, we can eliminate all numbers in the first and second columns after the second row. These Gauss method operations are equivalent to multiplying the matrix from the left by an invertible matrix.

If A is anti-symmetric, then it is easy to see that $BAB^T$ is also anti-symmetric. Let B be the matrix that is doing Gauss method operation to eliminate the first two columns under the top-left 2×2 block. Then $B^T$ does the same operations on the columns. Obviously, both B and $B^T$ are rational, so determinant is multiplied by a square of rational number. That number is nonzero, since B is invertible.

But now we get a block matrix, that consists of 2 anti-symmetric blocks, so the statement follows by induction over dimensions.

**Second proof.** It is known, that over anti-symmetric multi-linear forms the wedge product is defined, that makes a $k+m$-form out of $k$-form and m-form.

$$(\kappa \wedge \mu)(v_1, v_2, \ldots, v_m) = \frac{1}{k!m!} \sum_{\sigma \in \mathbb{S}_{k+m}} \operatorname{sgn}(\sigma)\, \kappa\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}\right) \cdot \mu\left(v_{\sigma(m+1)}, v_{\sigma(m+2)}, \ldots, v_{\sigma(m+k)}\right)$$

(here we divide by $k!m!$ to kill ambiguity – no need to sum equivalent summands several time, so this formula is actually integer).

This product is super-commutative and associative.

Any anti-symmetric 2-form can be represented in a general form as $\sum_{i<j} a_{ij} x_i \wedge x_j$, where $x_i$ are basic linear functionals corresponding to "taking $i$'th coordinate", or, when suitable basis is chosen, in a canonic form:
$\omega = k_1 x_1 \wedge x_2 + k_2 x_3 \wedge x_4 + k_3 x_5 \wedge x_6 + ... + k_n x_{2n-1} \wedge x_{2n}$.

Actually, that was what we have proven in the first solution.

But since the definition of the wedge product doesn't use coordinates, as well as some definitions of determinant, if we prove certain equality between those in the canonical basis, we shall know it for any basis.

Consider the product $\dfrac{\omega \wedge \omega \wedge ... \wedge \omega}{n!}$, where $\omega$ is multiplied by itself $n$ times.

When we open brackets, all products with similar factors cancel out. So we get $n!$ equivalent products, so after dividing by $n!$ we get an expression which is integer and not fractional in the coefficients, and that is $(k_1 k_2 k_3 ... k_n) x_1 \wedge x_2 \wedge x_3 \wedge ... \wedge x_{2n}$, product of all coefficients time standard volume form.

The determinant of the anti-symmetric matrix is $k_1^2 k_2^2 k_3^2 \cdot ... \cdot k_n^2$. It is the square of the coefficient before the volume form of $\dfrac{\omega \wedge \omega \wedge ... \wedge \omega}{n!}$. So it will be not necessarily in the canonical basis.

**Example.** Consider $n = 4$. Matrix $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$ is represented by a form $\omega = a_{12} x_1 \wedge x_2 + a_{13} x_1 \wedge x_3 + a_{14} x_1 \wedge x_4 + a_{24} x_2 \wedge x_4 + a_{23} x_2 \wedge x_3 + a_{34} x_3 \wedge x_4$.

Then $\dfrac{\omega \wedge \omega}{2} = (a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}) x_1 \wedge x_2 \wedge x_3 \wedge x_4$.

(When computing this things, just multiply each couple of terms once and don't divide by 2).

So $\det A = \left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)^2$.

**Outline of third solution** (Ofir Gorodetzky)

We know (either by guessing or from previous solution) the formula for the expression whose square is the determinant: it is a sum over all ways to decompose the set of all indices into pairs, of product of cells corresponding to that pairs (one index is of row, another of column), signs are chosen by the sign of a permutation which is formed when we write down all those pairs in a row, pair after pair.

So, we can prove combinatorially, that the square of that expression is the determinant. The determinant is a sum of all products over all permutations (or maximal rook arrangements). Some of those permutations contain odd cycles, others only even cycles. We can show that any permutation containing at least one odd cycle will cancel out with another permutation because the matrix is anti-symmetric (by transposing only that specific cycle).

So, we remain with permutations having even cycles only. Sides of even circle might be colored into black and white. That splits the permutation into two perfect matchings. Each of those perfect matchings can be considered as a summand in the polynomial we described, so the determinant is what we get after multiplying that expression by itself (since each time we unite 2 pair decompositions, we get a permutation with even cycles). Working out the signs is left as an exercise ☺.