Targil 12 – Analytic Geometry.

1. Consider segments AB, such that A is on *x* axis, B is on *y* axis, and length of AB is 1. The union of these intervals is a planar shape. Find an equation of the boundary of that shape.

Solution. Playing a bit with a pencil sliding along the edges of the desk shows you that it sweeps star-like area consisting of 4 symmetric concave triangles. The whole point is to find the envelope of that family of lines. AOB is a right triangle and AB = 1, we can take

$$A(\cos(t), 0)\ ,\ B(0, \sin(t))$$

So, let us take the intersection between two near intervals:

$$(\cos(t), 0)\ ,\ (0, \sin(t)) \text{ and } (\cos(t+\delta), 0)\ ,\ (0, \sin(t+\delta))$$

In general, the line passing via points $(a, 0)$ and $(0, b)$ has the equation $x/a + y/b = 1$ (it might remind the canonical form of equation of the ellipse equation). So, the two lines are

$$x / \cos(t) + y / \sin(t) = 1$$
$$x / \cos(t+\delta) + y / \sin(t+\delta) = 1$$

This is the same as

$$\begin{cases} \left(\frac{\sin(t+\delta)}{\cos(t+\delta)} - \frac{\sin(t)}{\cos(t)}\right)x = \sin(t+\delta) - \sin(t) \\ \left(\frac{\cos(t+\delta)}{\sin(t+\delta)} - \frac{\cos(t)}{\sin(t)}\right)y = \cos(t+\delta) - \cos(t) \end{cases}$$

$$\begin{cases} \frac{\sin(t+\delta-t)}{\cos(t+\delta)\cos(t)} \cdot x = \sin(t+\delta) - \sin(t) \\ \frac{-\sin(t+\delta-t)}{\sin(t+\delta)\sin(t)} \cdot y = \cos(t+\delta) - \cos(t) \end{cases}$$

When δ tends to 0, we get:

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases}$$

And that is parametric description of this curve.

From here we can get also the equation: $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 1$.

When we know the answer already, the solution can be made much shorter: simply compute the tangent and see that it cuts axes where it should.

Remark. This curve is called astroid. Notice that

$$\tfrac{1}{4}\left(\cos(3x)+3\cos(x)\right)=\cos^3(t)$$
$$\tfrac{1}{4}\left(\sin(3x)-3\sin(x)\right)=\sin^3(t)$$

So, another way to describe the astroid is as follows: a trajectory of a point on the boundary of a coin of radius 1/3 which is rolling inside the circular box of radius 1.
There is yet another unexpected way to describe the astroid: the locus of curvature centers of an ellipse.

2. For each t, take a line going through two points: $(t, 0)$ and $(0, 1 - t)$. When we draw all these line, part of the plane will be painted. Find a curve that separates the painted part of the plain from the unpainted.

First solution. Like before, we shall take two close lines and find their intersection point.
The line equations are:

$$\begin{cases} \dfrac{x}{t}+\dfrac{y}{1-t}=1 \\ \dfrac{x}{t+\delta}+\dfrac{y}{1-t-\delta}=1 \end{cases}$$

$$y\left(\frac{t+dt}{1-t-dt}-\frac{t}{1-t}\right)=dt$$

$$y\frac{(t+\delta)(1-t)-t(1-t-\delta)}{(1-t-\delta)(1-t)}=\delta$$

$$y\frac{\delta\cdot(1-t)+t\cdot\delta}{(1-t-\delta)(1-t)}=\delta$$

$$y\frac{\delta}{(1-t-\delta)(1-t)}=\delta$$

$$y=(1-t-\delta)(1-t)$$

When δ tends to 0, we get $y=(1-t)^2$.
There is a symmetry: we can replace x by y, and y by x, and t by $1 - t$, and get $x = t^2$. Alternatively, we can substitute the known value for y into the first equation, and get:

$$\frac{x}{t}+\frac{(1-t)^2}{1-t}=1$$

$$\frac{x}{t}+1-t=1$$

$$x=t^2$$

So, we have the parametric description: $\left(t^2,(1-t)^2\right)$.

It is tempting to write $\sqrt{x}+\sqrt{y}=t+(1-t)=1$. However, it is wrong. Indeed, the square root is the inverse of square only for positive numbers, so that equation only describes the arc of the curve when *t* and 1 – *t* are both nonnegative.
Also, that would be a sure way to get a contradiction in mathematics. The curve $\sqrt{x}+\sqrt{y}=1$ is contained in the square $[0,1]^2$ so it cannot touch, for instance, the line that goes via (3,0) and (0,–2).
Consider rotated coordinates:

$$\begin{cases} u=x-y=t^2-(1-t)^2=2t-1 \\ v=x+y=t^2+(1-t)^2=1-2t+2t^2 \end{cases}$$

Clearly, since $t = (u + 1)/2$, that line is a parabola.

So, the answer is: a parabola which is rotated by 45 degrees, and tangent to the axes at (0,1) and (1,0). It remains to check that the outer side of parabola is completely covered by our family of lines, and another isn't. It easy to see from the above computation, that the given family of lines is precisely the family of tangents to the parabola. The rest of it is an exercise (it follows from the convexity of parabola, and the fact that it doesn't have asymptotes).

Second solution. This solution is very simple, but I wouldn't find it if I wouldn't guess the answer first, which was noticed by Markelov.
It is based on the deep similarity between the circle and the parabola.

For example, compare the next two lemmas:
Lemma 1. Let A, B be two different points on a circle such that the lines PA, PB are tangent to the circle. Then PA = PB.
Lemma 2. Let A, B be two different points on a parabola $y = ax^2 + bx + c$ circle such that the lines PA, PB are tangent to the parabola. Then the projections of intervals PA, PB to the *x* axis are of the same length.

The lemmas are simple exercises (if you didn't know them yet). That similarity is deep: many geometric theorems about circles might be translated into theorems about aligned parabolas. During the translation, the distances must be replaced by the lengths of *x*–projections.

Consider parabola $y = ax^2 + bx + c$ and consider two points, A and B, such that the slope of tangent lines at those points is 45°. Let P be another point on the parabola. The tangent line at P intersects tangent lines at A

an B at points K and L, respectively, and tangent lines at A and B intersect at point T.
For each vector v, by v_x we shall denote its x projection. So, by lemma 2:

$$KL_x = KP_x + PL_x = AK_x + LB_x$$

But

$$KL_x + AK_x + LB_x = AB_x$$

So

$$KL_x = AK_x + LB_x = AB_x / 2$$

And from this the claim follows directly.

3.* We are given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A circle with center O is tangent to the ellipse externally (meaning they don't have internal common point); at the same time, there are two parallel lines tangent to both the circle and the ellipse. Find the locus of O satisfying these conditions.

Answer. A circle with center (0,0) and radius $a + b$.
Solution. The answer is easy to guess. When the pair of tangent lines is rotating, O goes around the 0 by a symmetric curve, which is definitely algebraic (since the conditions look algebraic) and probably of low degree. The vertical and horizontal pairs of tangent lines give yet another clue. So, now that we've guessed what to compute, let's do it.
Let T be the point of tangency between a circle and the ellipse.
Since it is on the ellipse, it can be written as $(a\cdot\cos(t), b\cdot\sin(t))$.
The gradient of the function $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ is $\left(\frac{2x}{a^2},\frac{2y}{b^2}\right)$. It is orthogonal to the level sets, one of which is an ellipse. So, the vector $\left(\frac{2\cos(t)}{a},\frac{2\sin(t)}{b}\right)$ is orthogonal to the ellipse at $(a\cdot\cos(t), b\cdot\sin(t))$. It is easy to see that vector looks outside the ellipse, so it is proportional to TO with positive coefficient. The same thing can be said about any parallel vector, for instance $(b\cdot\cos(t), a\cdot\sin(t))$. Therefore,

$$O = T + TO = (a\cdot\cos(t), b\cdot\sin(t)) + k(b\cdot\cos(t), a\cdot\sin(t))$$

Notice that if $k = 1$, we shall get $((a+b)\cdot\cos(t),(a+b)\cdot\sin(t))$, and that is the circle of radius $a + b$, which we have guessed already.
So, we should just check that there is a pair of common parallel tangents to the circle with center at that point and the ellipse.
Theoretically, on one ray TO orthogonal to the ellipse and directed outside, we could get more than one location of O. However, for each pair of parallel tangents at both directions there is only one circle. So, if

$\left((a+b)\cdot\cos(t),(a+b)\cdot\sin(t)\right)$ are solutions, then the pair of tangent lines rotate continuously all the way around the ellipse, and we cover all the possibilities. Therefore it is enough to check that those points satisfy the condition.
The pair of lines, symmetric with respect to 0 and tangent to the circle with center at O=$\left((a+b)\cdot\cos(t),(a+b)\cdot\sin(t)\right)$ is

$$x\cdot\sin(t)-y\cdot\cos(t)=\pm c,$$

since these are lines parallel to the line that goes via 0 and O.
Here *c* is the distance between 0 and those lines, because sum of the squares of coefficients of *x* and *y* is 1.
TO is also a radius of the circle, and it is equal $\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$.
So, it remains to verify that the lines

$$x\cdot\sin(t)-y\cdot\cos(t)=\pm\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$$

are tangent to the ellipse, or at least one of them (the other will follow from the symmetry).
Substitute $\left(a\cdot\cos(s),b\cdot\sin(s)\right)$ as (x,y).

$$a\cos(s)\cdot\sin(t)-b\sin(s)\cdot\cos(t)=\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$$

But by Cauchy-Schwartz inequality,

$$a\cos(s)\cdot\sin(t)-b\sin(s)\cdot\cos(t)\le$$

$$\le\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}\cdot\sqrt{\sin^2(s)+\cos^2(s)}=\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$$

So, the ellipse is one side of that line, and it touches it precisely once, when directions of vectors $\left(b\cos(t),a\sin(t)\right)$ and $\left(-\sin(s),\cos(s)\right)$ coincide. QED.

4.* Let P be a point upon the rectangular hyperbola $\{xy = 1\}$.
Let D be a symmetric point to P with respect to 0. Suppose a circle with center at P intersects the hyperbola $\{xy = 1\}$ at 4 points: A, B, C, D.
Prove that ABC is equilateral (regular) triangle.

Reminder. Each hyperbola has two asymptotes – straight lines that approximate it very well at all distant points. Hyperbola is called rectangular, if the asymptotes are orthogonal.

Solution. The following solution belongs to my high-school teacher, Dr. Anatoly Schulman.
Assume P = (u, v). The circle with center P is $(x - u)^2 + (y - v)^2 = R^2$.

A(x_A, y_A), B(x_B, y_B), C(x_C, y_C) and D($-u$, $-v$) belong to the circle and the hyperbola $y = 1/x$, so their x coordinate satisfies $(x - u)^2 + (1/x - v)^2 = R^2$. If we multiply by x^2 and expand it we shall get an equation of degree 4:

$$x^4 - 2u\,x^3 + k\,x^2 + m\,x + n = 0$$

Notice, that to each value of x only one point on hyperbola may correspond. So the four roots of this equation are precisely x_A, x_B, x_C, $-u$. Then by Vieta theorem, $x_A + x_B + x_C - u = 2u$.

$$x_A + x_B + x_C = 3u$$

Symmetric argument proves $y_A + y_B + y_C = 3v$.
So the mass center of triangle ABC is P, which is also its circumcenter. In other words, the meeting point of medians coincides with the meeting point of perpendicular bisectors of the sides. Thus the medians are the perpendicular bisectors, and hence the triangle ABC is equilateral.

5.** For a triangle ABC in plane, consider rectangular hyperbolas, going through A, B and C simultaneously. Each of those hyperbolas has a center of symmetry. Prove that all these centers lie on one circle.

Solution. First of all, let us understand how does an equation of a rectangular hyperbola look like. Equation of a conic is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Asymptotes are defined by intersection points with the infinite line. So, only the quadratic part, $ax^2 + bxy + cy^2$, influence the asymptotes. It can be decomposed as a product of linear equations, and those lines will be parallel to the asymptotes. If they are orthogonal then

$$ax^2 + bxy + cy^2 = k(mx + ny)(nx - my)$$

$$ax^2 + bxy + cy^2 = k(mnx^2 + (n^2 - m^2)y - mny^2)$$

It is easy to see that for given mn, the expression $(n^2 - m^2)$ may accept all values. So rectangular hyperbolas and couples of orthogonal lines (which are degenerate case of rectangular hyperbolas) are all the quadrics satisfying $a = -c$ and only them.
Consider now rectangular hyperbolas quadrics passing through A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3). They satisfy 4 linear equation. First 3 are

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

where i = 1, 2, 3. The last one is

$$a + c = 0$$

All those are linear equations in a, b, c, d, e, f. In 6-dimensional space 4 equation probably define 2-dimensional space, unless one of the equations is linear combination of the previous.
The second is not a multiple of the first, since it is easy to build a conic which contains A and doesn't contain B. It is also easy to find a conic containing A and B but not C. It is also easy to find a conic that contains A, B, C but isn't a rectangular hyperbola (for instance, circumcircle). So,

neither equation is a linear combination of the previous, and indeed we get a 2-dimensional linear space.
That space is spanned by each two non-proportional elements.
Actually, the space of our conics is better described a projective line, since multiplication of an equation by a constant doesn't alter the locus, described by the equation.

From here we can deduce a few conclusions, which are so nice that I cannot pass them by, even though they are not needed for the solutions.

(1) Three altitudes of triangle ABC have a common point. Indeed, consider an equation of quadric q_A which is a product of equations of two lines: BC and the altitude from A. Consider an equation of quadric q_B which is a product of equations of two lines: AB and the altitude from B. Define q_c in the similar way.
Let H be intersection point of altitudes from A and B. Then q_A and q_B have 4 common points: A, B, C, and H. But all rectangular hyperbolas in our family, and q_C among them, can be represented as $\lambda q_A + \mu q_B$, so they pass via A.
(2) Actually, we have generalized that elementary theorem about altitudes: all rectangular hyperbolas containing A, B, C also contain H, which is the orthocenter of triangle ABC.

Anyway, the equations of our rectangular hyperbolas are:

$$0 = ax^2 + bxy + cy^2 + dx + ey + f =$$
$$(a_0+\lambda a_1)x^2 + (b_0+\lambda b_1)xy + (c_0+\lambda c_1)y^2 + (d_0+\lambda d_1)x + (e_0+\lambda e_1)y + (f_0+\lambda f_1)$$

Now we need a way to compute the center of a hyperbola.
Apply parallel shift by (s, t) to the hyperbola. We get the equation

$$0 = a(x-t)^2 + b(x-t)(y-s) + c(y-s)^2 + d(x-t) + e(y-s) + f =$$
$$= ax^2 + bxy + cy^2 + (d - 2at - bs)x + (e - 2cs - bt)y + \mathrm{F}$$

(s, t) is the center of symmetry iff the equation became even. A condition for that is a pair of linear equation: linear coefficients are 0.

$$d - 2at - bs = 0$$
$$e - 2cs - bt = 0$$

In other words

$$2at + bs = d$$
$$bt + 2cs = e$$

The solution is

$$(4ac - b^2)t = 2cd - be$$
$$(4ac - b^2)s = 2ae - bd$$

So the center is

$$\left(\frac{2cd-be}{4ac-b^2},\frac{2ae-bd}{4ac-b^2}\right)=\left(\frac{-2ad-be}{-4a^2-b^2},\frac{2ae-bd}{-4a^2-b^2}\right)=\left(\frac{2ad+be}{4a^2+b^2},\frac{bd-2ae}{4a^2+b^2}\right)$$

The question is whether all those centers belong to one circle, i. e. whether they are described by one equation of the type

$$k(x^2+y^2) + lx + my + n = 0$$

$$\left(\frac{2ad+be}{4a^2+b^2}\right)^2+\left(\frac{bd-2ae}{4a^2+b^2}\right)^2=\frac{(2ad+be)^2+(bd-2ae)^2}{\left(4a^2+b^2\right)^2}=$$

$$=\frac{4a^2d^2+b^2e^2+4abde+b^2d^2+4a^2e^2-4abde}{\left(4a^2+b^2\right)^2}=$$

$$=\frac{4a^2d^2+b^2e^2+b^2d^2+4a^2e^2}{\left(4a^2+b^2\right)^2}=\frac{\left(4a^2+b^2\right)\left(d^2+e^2\right)}{\left(4a^2+b^2\right)^2}=\frac{d^2+e^2}{4a^2+b^2}$$

So, we need to find k, l, m, n such that:

$$k\frac{d^2+e^2}{4a^2+b^2}+l\frac{2ad+be}{4a^2+b^2}+m\frac{bd-2ae}{4a^2+b^2}+n=0$$

Which is the same as:

$$k\cdot\left(d^2+e^2\right)+l\cdot(2ad+be)+m\cdot(bd-2ae)+n\cdot\left(4a^2+b^2\right)=0$$

Since the possible values of a, b, d, e, f are linear expressions in λ, hence those brackets are quadratic expressions in λ:

$$k\cdot\left(p_0+p_1\lambda+p_2\lambda^2\right)+l\cdot\left(q_0+q_1\lambda+q_2\lambda^2\right)+$$
$$+m\cdot\left(r_0+r_1\lambda+r_2\lambda^2\right)+n\cdot\left(s_0+s_1\lambda+s_2\lambda^2\right)=0$$

So, it is enough to find non-zero solution to the system of 3 homogenous equations: $p_ik+q_il+r_im+s_in=0$, for $i = 0, 1, 2$. These equations have nontrivial solution, hence the centers are on one line or circle.
But the foots of altitudes are not one line, so they are on one circle.

Remark. This circle is famous: it is called Euler's circle, Feuerbach's circle, and nine-point circle. The nine point are: midpoints of the 3 sides, midpoints of the intervals AH, BH, CH where H is the orthocenter, and the foots of the three altitudes. The fact that those nine points are on one circle is considered one of the gems of the elementary geometry. Problem 5 gives a generic description for all points of the nine-point circle, and not just for 9 of them.

6. ABCD is a tetrahedron in the space. For each edge, consider plane passing via its midpoint and orthogonal to the opposite edge (for instance, a plane via the middle of AB orthogonal to CD). Prove that these 6 planes intersect in one point.

Remark. This point is called Monge point.

Solution. The uniqueness of that point is trivial (otherwise all planes would be parallel to one line, but then all edges of ABCD would be parallel to one plane, then ABCD would be a planar shape and not tetrahedron). We are looking for the point M, such that $M - (A+B)/2$ is orthogonal to $C - D$, along with all the symmetric conditions.
Try $M = (A + B + C + D) / 2$, then $M - (A+B)/2 = (C + D)/2$.
Then we want to require $0 = (C - D , (C + D)/2)$. That is the same thing as $0 = (C - D , C + D) = (C , C) - (D , D)$.
So, if we have chosen the origin to be the center of circumsphere of ABCD, then $|A| = |B| = |C| = |D|$ and it will just work.