Targil 11 – once again, linear algebra

1. Let $A_1A_2\ldots A_n$ be a polygon inscribed in circle. Consider a skew symmetric $n\times n$ matrix (a_{ij}), such that for $i < j$, $a_{ij} = A_iA_j$. Prove that the rank of this matrix is not greater than 2.

2. Let A, B, C be $n\times n$ square matrices. Prove that

$$\mathrm{rk}(AB) + \mathrm{rk}(BC) \leq \mathrm{rk}(ABC) + \mathrm{rk}(B).$$

3.* **(a)** A linear operator A over $\mathbb{C}^n$ can be considered as a linear operator A_r over $\mathbb{R}^{2n}$, because $\mathbb{C}^n$ is a $2n$-dimensional space over $\mathbb{R}$. Prove that $|\det(A)|^2 = \det(A_r)$.

(b) Formulate and prove a more general claim, about finite field extension (field $\mathbb{C}$ is an extension of field $\mathbb{R}$ of degree 2).

4. Consider matrix equation $AX - XB = C$, where A, B, C are given $n\times n$ matrixes, and X is an unknown $n\times n$ matrix. Show that the solution of the equation exists and unique if and only if A and B don't have a common eigenvalue.

5.* Let A be an invertible $n\times n$ real matrix, U, V be linear subsets of $\mathbb{R}^n$. Assume that U and V are ***almost disjoint***, which means they have no more common elements except 0.
Show that there exists an integer k such that A^kU and V are almost disjoint.