Selection of Israeli Team for IMC 2008.

1. A group is generated by two elements a,b. The following relations hold:

$$a^2 = 1$$
$$b^2 = 1$$
$$(ab)^{10} = 1$$

Find the maximal possible size of this group.

Answer. 20.

Solution. $a^{-1} = a$, $b^{-1} = b$, hence every element in the group can be expressed as an a product of a's and b's .

But $ab\ ab\ ab\ ab\ ab\ ab\ ab\ ab\ ab\ a = b$, hence every element can be expressed as a word starting with a. Represent each element by the shortest word starting with a. It cannot have aa or bb in the middle, otherwise we could make it shorter. So, a's and b's alternate in this word. So the word is $abab\ldots$ and it is determined by its length. We cannot have more than 20 symbols, since they would cancel out. So there are no more, than 20 elements in the group.

Now let's see an example of such a group with precisely 20 elements. Consider the group of symmetries of regular 10-gon. There are exactly 20. Let a be a mirror reflection with respect to an orthogonal bisector of two opposite sides, and b be a mirror reflection with respect to a diagonal, passing through two opposite vertices of those sides. Then ab is a rotation by 36^o hence $(ab)^{10} = 1$. Since a, b are reflections $a^2 = b^2 = 1$. It is easy to see that the group is generated by a and b, since ab generates all rotations, it also has 2 reflections, so we have more than 10 elements of the group generated by a and b. The order of subgroup generated by a, b, is a divisor of 20 which is bigger than 10, so it is 20.

2. For $f : \mathbb{R} \to \mathbb{R}$ it is given that any positive real $x,\ y$ the sequence $f(x+ny)$,

for $n \in \mathbb{N}$, tends to infinity. Does it follow that $f(x) \to \infty$ as $x \to \infty$?

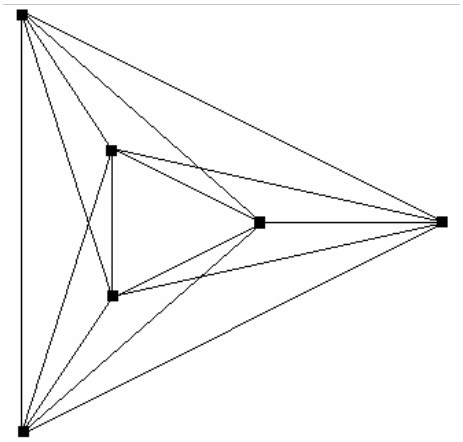
Answer. No.

Solution. Let $a > 1$ be a transcendent number, and consider the following function: $f(a^n) = 0$ for every natural n, and $f(x) = x^2$ for all other points. Any arithmetic progression can have no more than two common points with the sequence a^n, since if it would have 3 common points, a would be a root of a polynomial with rational coefficients. Hence any sequence $f(x+ny)$ tends to infinity, and $f(x)$ doesn't.

3. Denote $[X,Y] = XY - YX$ (it is called commutator). Assuming that A, B, C are 2×2 matrices, prove that $\left[[A,B]^2, C\right] = 0$.

Solution. tr XY = tr YX. Hence tr($XY - YX$) = 0.
Consider matrix $[A,B]$ in its Jordan form. It has either one Jordan cell 2×2 or two Jordan cells 1×1. Trace = 0, so in the first case this matrix is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and in the second case $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. In both cases the $[A,B]^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, so it commutes with any C.

4. For which N can we draw a full graph of N vertices on the plane, so that each arc will be intersected no more than once, and no three arcs would have a common inner point?



Solution. $N < 7$.
The picture shows a graph for $N = 6$, such pictures for smaller N can be obtained by erasing points from this picture.
Suppose we have a full graph for 7 vertices, which satisfies all conditions. It has 21 edges, and X crossings between edges.
Let O be a crossing point, between arcs AC and BD. Consider quadrangle defined by arcs AB, BC, CD, and DA. We can assume that its edges are uncrossed, because if an edge would enter triangle ABO by crossing AB, it will have to exit the triangle through AB, AO, or BO and make second crossing with an edge which is crossed already. Unless the edge enters ABO through AB and stops at A or at B, but then we can shift the edges a bit and eliminate that intersection.
So, if we draw a graph with minimal possible number of crossings, then around each crossing we shall have 4 uncrossed edges. Here each uncrossed edge can be counted at most twice, hence the number of uncrossed edges is at least 2X. The number of crossed edges is precisely 2X, so $4X < 21$, hence $X \le 5$.

Consider a graph whose vertices are points of crossing together with the original vertices, and edges are original uncrossed edges and halves of crossed edges. This graph, unlike the original, is planar. It has V=7+X vertices, E = 21+2X edges, and $F \le {}^2\!/_3 E$ faces hence by Euler's formula

$$2 = F - E + V \le V - E/3 = 7 + X - (7 + 2X/3) = X/3$$

Hence $6 \le X$.
But before we proved $X \le 5$, a contradiction.
Since we cannot build such a graph for 7, we can't do it for higher *N* either.

5. For tetrahedron ABCD, an altitude (גובה) is a straight line passing through one vertex and orthogonal to the plane containing 3 other vertexes.
Prove that there exists a straight line, passing through vertex A and having a common point with each tetrahedron's altitude.

Solution. Assume A is (0,0,0), denote vectors u = AB, v = AC, w = AD. Normal vector to the plane ABC is $[u,v]$, the vector product, each vector x in that plane is orthogonal to $[u,v]$. The plane, passing through A, and containing the altitude from D, is contains vectors w and $[u,v]$, so it's normal vector is $[[u,v],w]$, (vectors w and $[u,v]$ are parallel, since it is given, that w is not orthogonal to u and v). Hence any line, passing through A and orthogonal to $[[u,v],w]$, intersects the altitude from D, unless they are parallel. So, we need to have a line which is orthogonal to $[[u,v],w]$ and two similar vectors $[[w,u],v]$ and $[[v,w],u]$, but at the same time, that line should not be parallel to any of altitudes from B, C, D, which are parallel to the vectors $[u,v]$, $[w,u]$, $[v,w]$. Recall Jacobi identity:

$$[[u,v],w] + [[w,u],v] + [[v,w],u] = 0$$

If You don't, prove it by applying thrice a well known formula of analytic geometry:

$$[[u,v],w] = (u,w)v - (v,w)u$$

From Jacobi identity, it follows that vectors $[[u,v],w]$, $[[w,u],v]$, $[[v,w],u]$ really are coplanar and have a common orthogonal vector. It still remains to prove that this vector can't be, say, $[u,v]$.
Assume that $[u,v]$ is orthogonal to $[[v,w],u]$.
Then there are real numbers a, b such that

$$a[v,w] + bu = [u,v]$$

Take scalar product with v, and we get

$$b(u,v) = 0$$

But u isn't orthogonal to v, hence $b = 0$, so $a[v,w] = [u,v]$, so normal vectors of two different faces are parallel, so two different faces are parallel, which is impossible.