Israeli Team for SEEMOUS
Final Selection Exam

Please write your solutions in English.

1. A ***graph*** is, by definition, a collection of ***vertices*** and a collection of ***edges*** that connect pairs of vertices. Two vertices are called ***adjacent***, if they share an edge.
Given a graph, consider the function $c(n)$ – the number of ways to color each vertex with one of n given colors, so that no two adjacent vertices will have the same color. Show, that $c(n)$ is a polynomial of n.

2. A disc of radius $1/N$ is rolling inside the circular box of radius 1, where $N > 2$. (The friction between the edge of the disc and the wall of the box is very high so the disc doesn't slip with respect to the box at the point of tangency). A red point on the boundary of a small circle goes along a star-shaped closed trajectory.
Compute the area, bounded by this trajectory (as a function of N).

3. A natural number k is considered ***good***, if for each N the number $1^k+2^k+\ldots N^k$ is divisible by $1+2+\ldots+N$.
Describe the set of all good numbers.

4. Let $\boldsymbol{A_1, A_2,\ldots,A_N}$ be nonzero matrices $M\times M$ (a matrix is called ***nonzero*** if at least one of its elements is nonzero). Prove that there exists a matrix B of the same size such that $\boldsymbol{BA_1BA_2B\ldots BA_NB}$ is a nonzero matrix.

5. An infinite sequence of real numbers $\{x_i\}$ will be called ***nice*** if $\sum x_i^2$ converges. Let $\{a_i\}$ be a sequence, such that for each nice sequence $\{x_i\}$ the series $\sum a_i x_i$ converges. Prove that the sequence $\{a_i\}$ is nice.

Good luck!